

SEMICLASSICAL TRACE FORMULA FOR THE 2-DIMENSIONAL RADIAL POWER-LAW POTENTIALS

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Abstract

The trace formula for the density of single-particle levels in the two-dimensional radial power-law potentials, which nicely approximate the radial dependence of the Woods-Saxon potential and quantum spectra in a bound region, was derived by the improved stationary phase method. The specific analytical results are obtained for the powers 4 and 6. The enhancement phenomena near the bifurcations of periodic orbits are found to be significant for the description of the fine shell structure. It is shown that the semiclassical trace formulas for the shell corrections to the level density and energy reproduce the quantum results with good accuracy through all the bifurcation (symmetry breaking) catastrophe points, where the standard stationary-phase method breaks down. Various limits (including the harmonic oscillator and the spherical billiard) are obtained from the same analytical trace formula.

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I. INTRODUCTION

According to the shell-correction method (SCM) [1, 2], the oscillating part of the total energy of a finite fermion system, the so-called shell-correction energy δU , is associated with an inhomogeneity of the single-particle energy levels near the Fermi surface. Depending on the level density at the Fermi energy – and with it the shell-correction energy δU – being a maximum or a minimum, the nucleus is particularly unstable or stable, respectively. This situation varies with particle numbers and nuclear-potential parameters like deformation ones. In consequence, the stability of nuclei depend strongly on particle numbers and parameters of the potential.

A semiclassical theory of shell effects, the so-called periodic orbit theory (POT) [3–7], was used for a deeper understanding, based on classical pictures, of the origin of nuclear shell structure and its relation to a possible chaotic nature of the dynamics of nucleons. This theory provides us with a nice tool for answering, sometimes even analytically, the fundamental questions concerning the physical origins of the double-humped fission barrier and, in particular, of the existence of the isomer minimum [8–13] (see Refs. [12, 13] for the introductory review and text book). Some applications of the POT to nuclear deformation energies were presented and discussed for the cavity (billiard-like) potentials with sharp edges in relation to the bifurcations of periodic orbits with the pronounced shell effects.

In the way to more realistic semiclassical calculations, it is important to account for a diffuseness of the nuclear edge. It is known that the central part of the realistic effective mean-field potential for nuclei or metallic clusters are described by the Woods-Saxon (WS) potential $V_{\text{WS}}(r)$ [14]. The idea of Refs. [15, 16] is that the WS potential is nicely approximated (up to a constant shift) by much a simpler power-law potential which is proportional to a power of the radial coordinate r^α . The approximate equality

$$V_{\text{WS}}(r) \approx V_{\text{WS}}(0) + W_0 r^\alpha \quad (1.1)$$

holds up to around the Fermi energy with a suitable choice of the parameters W_0 and α . In the case of the spatial dimension $\mathcal{D} = 2$, one can use Eq. (1.1) for a realistic potential of electrons in a circular quantum dot [12, 13, 17]. We shall derive first the generic trace formula for this radial power-law (RPL) potential in the case of two dimensions, and then discuss its well known limits to the harmonic oscillator and cavity (billiard) potentials [13].

The manuscript is organized as follows. In Sec. II the classical dynamics is specified for the RPL potentials. The outlook of the general phase-space trace formula for arbitrary potentials is presented in Sec. III. The derivations of the trace formulas for the RPL potentials in two dimensions is given in Sec. IV. Section V is devoted to the comparison of the semiclassical calculations for the oscillating level density, and the shell-correction energy with quantum results. The paper is summarized in Sec. VI. Some details of our POT calculations, in particular full analytical derivations at the powers $\alpha = 4$ (see also Ref. [18]) and 6 for all periodic orbits (POs) and those at arbitrary α for the diameter and circle orbits, are given in the Appendixes A–E.

II. CLASSICAL DYNAMICS AND BIFURCATIONS

The radial power-law (RPL) potential model is described by the Hamiltonian

$$H = \frac{p^2}{2m} + E_0 \left(\frac{r}{R_0} \right)^\alpha, \quad (2.1)$$

where m is the mass of the particle, R_0 and E_0 are introduced as constants, having the dimension of length and energy, respectively, and are related with W_0 in Eq. (1.1) by $W_0 = E_0/R_0^\alpha$. (Thus, E_0 can be arbitrary, and it is convenient to take $E_0 = \hbar^2/mR_0^2$ in quantum mechanical formulation.) This Hamiltonian includes the limits of the harmonic oscillator ($\alpha = 2$) and the cavity ($\alpha \rightarrow \infty$); realistic nuclear potentials with steep but smooth surfaces correspond to values in the range $2 < \alpha < \infty$. The advantage of this potential is that it is a homogeneous function of the coordinates, so that the classical equations of motion are invariant under the scale transformations:

$$\begin{aligned} \mathbf{r} \rightarrow s^{1/\alpha} \mathbf{r}, \quad \mathbf{p} \rightarrow s^{1/2} \mathbf{p}, \quad t \rightarrow s^{1/\alpha-1/2} t \\ \text{with } E \rightarrow sE. \end{aligned} \quad (2.2)$$

Therefore, one only has to solve the classical dynamics once at a fixed energy, e.g., $E = E_0$ ($s = 1$); the results for all other energies E are then simply given by the scale transformations (2.2) with $s = E/E_0$ by definition in the last equation of Eq. (2.2). This highly simplifies the POT analysis [15, 19]. Note that the definition (2.1) can also be generalized to include deformations (see, e.g., Ref. [16, 19]).

As the RPL Hamiltonian H is spherically symmetric, it can be written explicitly in the two-dimensional (2D) spherical canonical phase-space variables $\{r, \varphi; p_r, p_\varphi\}$,

$$H = \frac{1}{2m} \left(p_r^2 + \frac{p_\varphi^2}{r^2} \right) + E_0 \left(\frac{r}{R_0} \right)^\alpha = E, \quad (2.3)$$

where φ is the azimuthal angle (a cyclic variable), $p_\varphi = L$ is the angular momentum, and the radial momentum p_r is given by

$$p_r(r, L) = \sqrt{p^2(r) - \frac{L^2}{r^2}},$$

$$p(r) = \sqrt{2m \left[E - E_0 \left(\frac{r}{R_0} \right)^\alpha \right]}. \quad (2.4)$$

The classical trajectory (CT) $r(t)$ can be easily found by integrating the radial equation of motion $\dot{r} = p_r/m$ with Eq. (2.4). Transforming the spherical canonical variables into the action-angle ones, for the actions I_r, I_φ one has

$$I_r = \frac{1}{\pi} \int_{r_{\min}}^{r_{\max}} p_r dr \equiv I_r(E, L), \quad (2.5)$$

$$I_\varphi = \frac{1}{2\pi} \int_0^{2\pi} p_\varphi d\varphi \equiv L, \quad (2.6)$$

where r_{\min} and r_{\max} are the turning points which are the two real (positive) solutions of the equation $p_r^2(r, L) = 0$.

The definition (2.1) can be used in arbitrary spatial dimensions, as long as r is the corresponding radial variable. In practice, we are interested only in the 2D and 3D cases. The spherical 3D and the circular 2D potential models have common sets of periodic orbits, see Fig. 1. For $\alpha > 2$, POs with the highest degeneracy ($\mathcal{K} = 1$ and 3 in the 2D and 3D cases, respectively) are specified by three integers and labeled as $M(n_r, n_\varphi)$, where n_r and n_φ are mutually commensurable integers. They represent the number of oscillations in the radial direction (n_r), and the number of rotations around the origin (n_φ) for one period, and M is the repetition number. For the isotropic harmonic oscillator ($\alpha = 2$), all the classical orbits are periodic ones with (degenerate) ellipse shapes. By slightly varying α away from 2, the specific diameter and circle orbits appear separately, and they remain as the shortest POs with the corresponding degeneracies $\mathcal{K} = 1$ and 0. With increasing α , the circle orbit and its repetitions cause successive bifurcations generating various new periodic orbits $\{n_r, n_\varphi\}$, $n_r > 2n_\varphi$. Fig. 1 shows some of the shortest POs $M(n_r, n_\varphi)$. The shortest

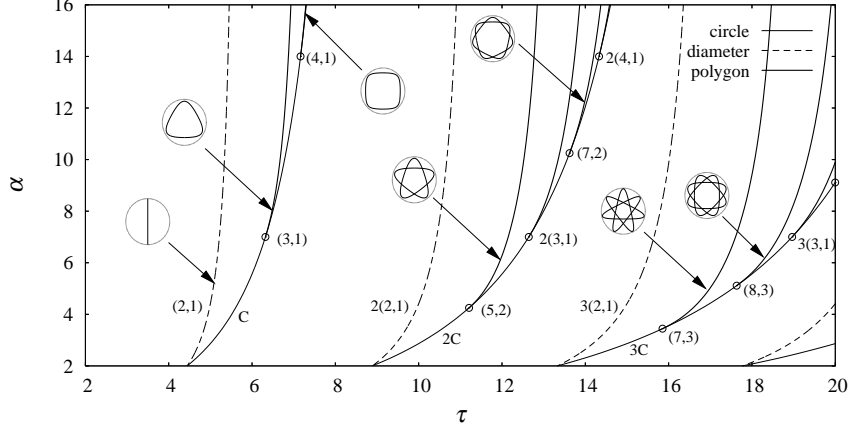


FIG. 1. The scaled periods τ_{PO} of some short periodic orbits, plotted as functions of the power parameter α in dimensionless units $m = R_0 = E_0 = 1$ (Appendix B). Thin solid curves are the circle orbits MC , dashed curves are the diameters $M(2,1)$, and thick solid curves are the polygon-like orbits $M(n_r, n_\varphi)$ ($n_r > 2n_\varphi$), which bifurcate from the circle orbits MC at the bifurcation points indicated by open circles.

PO is the diameter which has the degeneracy $\mathcal{K} = 1$ in the 2D problem at $\alpha > 2$. Other polygon-like orbits have $\mathcal{K} = 1$ at $\alpha > \alpha_{\text{bif}}$, where α_{bif} is a bifurcation value (see its specific expression below). The circle orbit having maximum angular momentum is isolated ($\mathcal{K} = 0$) for the 2D system (except for the bifurcation points).

For the frequencies of the radial and angular motion of particle, one finds

$$\begin{aligned} \omega_r &= \frac{\partial H}{\partial I_r} = \left(\frac{\partial I_r}{\partial E} \right)_L^{-1} = \left(\frac{m}{\pi} \int_{r_{\min}}^{r_{\max}} \frac{dr}{p_r(r, L)} \right)^{-1}, \\ \omega_\varphi &= \frac{\partial H}{\partial L} = - \frac{(\partial I_r / \partial L)_E}{(\partial I_r / \partial E)_L}, \end{aligned} \quad (2.7)$$

where $p_r(r, L)$ is the radial momentum, given in Eq. (2.4), and $I_r = I_r(E, L)$ [Eq. (2.5)] is identical to the energy surface $H(I_r, L) = E$. Thus, the PO condition is written as

$$\begin{aligned} f(L) &\equiv \frac{\omega_\varphi}{\omega_r} = - \left(\frac{\partial I_r(E, L)}{\partial L} \right)_E \\ &= \frac{L}{\pi} \int_{r_{\min}}^{r_{\max}} \frac{dr}{r^2 p_r(r, L)} = \frac{n_\varphi}{n_r}. \end{aligned} \quad (2.8)$$

The energy surface $I_r = I_r(E, L)$ is simplified to a function of the only one variable L [see Eq. (2.5)]. The solutions to the PO equation (2.8), $L^* = L^*(n_r, n_\varphi)$, for the given co-primitive integers n_φ and n_r define the one-parametric families $\mathcal{K} = 1$ of orbits $M(n_r, n_\varphi)$

because L is the single-valued integral of motion, which is only 1, besides of the energy E . [Notice that another single-valued integral of motion, $Q = \sin(n_r\Theta_\varphi - n_\varphi\Theta_r)$, appears under the PO condition (2.8) (Θ_r and Θ_φ are the angles of the action-angle variables). However, for these RPL potentials, no additional degeneracy arises since the rationality of frequencies occurs only at the discrete values of angular momenta L^* ; see Refs. [5, 8].] The azimuthal angle φ can be taken, for instance, as a parameter of the orbit of such a family.

According to the limit $f(L) \rightarrow 1/2$ at $L \rightarrow 0$, one has the diameter orbits $M(2, 1)$ as the specific one-parametric ($\mathcal{K} = 1$) families related to the solution $L = 0$ of Eq. (2.8). The other specific solutions are the isolated ($\mathcal{K} = 0$) circle orbits MC by which we represent the M -th repetition of the primitive circle orbit C . The radius r_c of the circle orbit is determined by the system of equations, namely $r_{\min} = r_{\max} \equiv r_c$ for equivalence of the turning points being, in turn, the solutions of the equation $p_r^2(E, L) = 0$ (see Appendix A). Thus, one has also the angular momentum of the circle orbit $L_C = r_c p(r_c)$. As seen obviously from the condition of the real radial momentum p_r [Eq. (2.4)], this L_C is the maximal value of the angular momentum L , i.e., $0 \leq |L| \leq L_C$.

As shown in Appendix A, for the calculation of the Gutzwiller stability factor F_{MC} of the circle orbit MC in the radial direction, one obtains

$$\begin{aligned} F_{MC} &= 2 - \text{Tr}(\mathcal{M}_C)^M = 4 \sin^2 \left[\frac{\pi M \Omega_C}{\omega_c} \right] \\ &= 4 \sin^2 \left[\pi M \sqrt{2 + \alpha} \right], \end{aligned} \quad (2.9)$$

where Ω_C/ω_c is the ratio of the radial-to-angular frequencies,

$$\frac{\Omega_C}{\omega_c} = \sqrt{2 + \alpha}, \quad \omega_c = \omega_\varphi(L = L_C) = \frac{L_C}{m r_c^2}. \quad (2.10)$$

The factor F_{MC} becomes zero at the bifurcation points α_{bif} by the definition of the stability matrix $\text{Tr}(\mathcal{M}_C)^M = 2$; see Eq. (2.9). According to the first equation in Eq. (2.10), one finds

$$\alpha_{\text{bif}} = \frac{n_r^2}{n_\varphi^2} - 2 \quad (2.11)$$

for the POs, $\Omega_C/\omega_c = n_r/n_\varphi$. The PO family $M(n_r, n_\varphi)$, which corresponds to the solutions $L^* < L_C$ of the PO equation (2.8), exists for all $\alpha > \alpha_{\text{bif}}$. There is one specific bifurcation point $\alpha = 2$ in the spherical harmonic oscillator (HO) limit with the frequency $\omega_\varphi = \sqrt{2E_0/(mR_0^2)}$, where one has the two-parametric families at any L within a continuum $0 \leq L \leq E/\omega_\varphi$. In this limit the above specified circle and diameter orbits belong

to a two-parametric family mentioned above. For the circular billiard limit $\alpha \rightarrow \infty$, the isolated circle orbit ($\mathcal{K} = 0$) disappears degenerating into the billiard boundary $r_c \rightarrow R_0$, $L_C \rightarrow \sqrt{2mE} R_0$ [see the limit $\alpha \rightarrow \infty$ in Eq. (A2) for r_C].

Another key quantity in the POT is the curvature K of the energy surface $I_r = I_r(E, L)$ given by

$$K = \frac{\partial^2 I_r(E, L)}{\partial L^2} = -\frac{\partial f(L)}{\partial L}, \quad (2.12)$$

where $f(L)$ is the ratio of frequencies defined in Eq. (2.8). As shown below, the curvature (2.12) and Gutzwiller factor (2.9), related both to the stability of the orbit, are the key quantities for calculations of the magnitude of the PO contributions into the semiclassical level density.

III. PHASE-SPACE TRACE FORMULA

The level density $g(E)$ for the Hamiltonian $H(\mathbf{r}, \mathbf{p})$ can be obtained by using the phase-space trace formula (in \mathcal{D} dimensions) [10, 11, 20–23]:

$$g_{\text{scl}}(E) = \frac{1}{(2\pi\hbar)^{\mathcal{D}}} \text{Re} \sum_{\text{CT}} \int d\mathbf{r}' \int d\mathbf{p}'' \delta(E - H(\mathbf{r}'', \mathbf{p}'')) \\ \times |\mathcal{J}_{\text{CT}}(\mathbf{p}'_{\perp}, \mathbf{p}''_{\perp})|^{1/2} \exp\left(\frac{i}{\hbar} \Phi_{\text{CT}} - i\frac{\pi}{2} \mu_{\text{CT}}\right). \quad (3.1)$$

The sum is taken over all discrete CT manifolds for a particle moving between the initial \mathbf{r}', \mathbf{p}' ; and the final $\mathbf{r}'', \mathbf{p}''$ points with a given energy E . Any CT can be uniquely specified by fixing, for instance, the initial condition \mathbf{r}' , and the final momentum \mathbf{p}'' for a given time t_{CT} of the motion along the CT. For the action phase Φ_{CT} in exponent of (3.1), one has

$$\Phi_{\text{CT}} \equiv S_{\text{CT}}(\mathbf{p}', \mathbf{p}'', t_{\text{CT}}) + (\mathbf{p}'' - \mathbf{p}') \cdot \mathbf{r}' \\ = S_{\text{CT}}(\mathbf{r}', \mathbf{r}'', E) - \mathbf{p}'' \cdot (\mathbf{r}'' - \mathbf{r}'), \quad (3.2)$$

where $S_{\text{CT}}(\mathbf{p}', \mathbf{p}'', t_{\text{CT}})$ and $S_{\text{CT}}(\mathbf{r}', \mathbf{r}'', E)$ are the actions in the momentum and coordinate representations, respectively,

$$S_{\text{CT}}(\mathbf{p}', \mathbf{p}'', t_{\text{CT}}) = - \int_{\mathbf{p}'}^{\mathbf{p}''} d\mathbf{p} \cdot \mathbf{r}(\mathbf{p}), \\ S_{\text{CT}}(\mathbf{r}', \mathbf{r}'', E) = \int_{\mathbf{r}'}^{\mathbf{r}''} d\mathbf{r} \cdot \mathbf{p}(\mathbf{r}). \quad (3.3)$$

In Eq. (3.1), $\mathcal{J}_{CT}(\mathbf{p}'_{\perp}, \mathbf{p}''_{\perp})$ is the Jacobian for the transformation of the initial perpendicular-to-CT momentum \mathbf{p}'_{\perp} to the final one \mathbf{p}''_{\perp} . μ_{CT} is the Maslov phase related to the number of conjugate (turning and caustics) points along the CT [24–26].

For calculations of the trace integral (3.1) by the stationary phase method (SPM), one may write the stationary phase conditions in both \mathbf{p}'' and \mathbf{r}' variables. According to the definitions (3.2) and (3.3), the stationary phase condition for the \mathbf{p}'' variable is that for the CT to be closed in the coordinate space:

$$\left(\frac{\partial \Phi_{CT}}{\partial \mathbf{p}''} \right)^* \equiv (\mathbf{r}' - \mathbf{r}'')^* = 0. \quad (3.4)$$

Here and below, the star means that any quantities (as in the parentheses) are taken at the stationary point; in particular, $\mathbf{p}'' = \mathbf{p}''^*$ in the variable \mathbf{p}'' . In the next integration over \mathbf{r}' by the SPM, we use the Legendre transformation (3.2), and the closing condition (3.4) which lead to $\Phi_{CT} = S_{CT}(\mathbf{r}', \mathbf{r}'', E)$. Therefore, the stationary phase condition for this integration over the spatial coordinates $\mathbf{r} = \mathbf{r}' = \mathbf{r}''$ can be written in the following forms:

$$\begin{aligned} \left(\frac{\partial \Phi_{CT}}{\partial \mathbf{r}'} + \frac{\partial \Phi_{CT}}{\partial \mathbf{r}''} \right)^* &\equiv \left(\frac{\partial S_{CT}}{\partial \mathbf{r}'} + \frac{\partial S_{CT}}{\partial \mathbf{r}''} \right)^* \\ &\equiv -(\mathbf{p}' - \mathbf{p}'')^* = 0, \end{aligned} \quad (3.5)$$

where the star means $\mathbf{r}' = \mathbf{r}'' = \mathbf{r}^*$ along with $\mathbf{p}' = \mathbf{p}'' = \mathbf{p}^*$. Therefore, one has the condition for a CT to be closed also in the momentum space, and then, the *stationary phase* conditions are equivalent to the *periodic-orbit* equations (3.4) and (3.5). After applying these two conditions we may arrive at the trace formula in terms of the sum over POs [12, 13].

IV. TRACE FORMULA FOR THE 2D POWER-LAW POTENTIALS

As showed in Sec. II, there are two kind of POs in the 2D circular potential; i.e., one-parametric families ($\mathcal{K} = 1$, including the diameters), and isolated ($\mathcal{K} = 0$) circle POs; for which we shall derive the trace formulas separately in the subsections A and B, respectively. In these derivations for both kinds of POs, it is useful to take the canonical integration variables with the angular momentum L as one of the integration variables in the phase-space trace formula (3.1). As shown in the subsection A for the contributions of the one-parametric ($\mathcal{K} = 1$) families of the diameters and polygon-like POs, the integration over the cyclic angle variable φ , conjugate to L , is easily carried out (it simply gives 2π). The remaining

integration over L is done within the simplest improved SPM (ISPM [11, 12, 22, 23]) with the finite limits $-L_C \leq L \leq L_C$. Thus, we shall obtain the ISPM trace formula in terms of error functions [12]. Extending the integration limits to $\pm\infty$ far from the critical symmetry-breaking points, we will arrive at the asymptotic Berry-Tabor trace formula [6] for the contributions of P (polygon-like) and D (diameter) POs. In the next subsection B, the contribution of the isolated ($\mathcal{K} = 0$) circle orbits is derived by adopting the ISPM for two integrations over suitable (radial phase-space) variables. It yields a product of two error functions. Extension of the finite integration limits in both error functions to infinity (i.e., to $\pm\infty$ after their transformation to the corresponding Fresnel integrals of positive or negative real arguments) leads to the standard SPM (SSPM) Gutzwiller trace formula [3] for the considered isolated POs [3].

A. One-parametric orbit families ($\mathcal{K} = 1$)

For the contribution of the one-parametric *families* of the maximal degeneracy $\mathcal{K} = 1$ into the phase-space trace formula (3.1), it is useful to transform canonically the usual Cartesian phase-space variables $\{\mathbf{p}; \mathbf{r}\}$ to the action-angle ones $\{\mathbf{I}; \boldsymbol{\Theta}\}$, specified in the spherical action-angle variables as $\boldsymbol{\Theta} = \{\Theta_r, \Theta_\varphi \equiv \varphi\}$; $\mathbf{I} = \{I_r, I_\varphi \equiv L\}$. The Hamiltonian H , action phase Φ_{CT} , and other related quantities of the integrand in Eq. (3.1) are functions only of action variables [$H = H(\mathbf{I}) = H(I_r, I_\varphi) \equiv H(I_r, L)$], i.e., they are independent of the angle variables $\boldsymbol{\Theta}$. Therefore, performing easily the integrations over these angle variables $\boldsymbol{\Theta}$, one has the factor 2π per each integration. Taking then the integral over I_r exactly by using the energy conserving δ -function, one obtains

$$g_{\text{scl}}(E) = \frac{1}{2\hbar^2} \text{Re} \sum_{M, n_r, n_\varphi} \int dL \frac{1}{\omega_r} \times \exp \left\{ \frac{2\pi i}{\hbar} M [n_r I_r(E, L) + n_\varphi L] - \frac{i\pi}{2} \mu_{M, n_r, n_\varphi} \right\}. \quad (4.1)$$

Here, the phase (3.2) is expressed in terms of the corresponding action-angle variables through the actions (3.3) in the considered mixed representation,

$$\Phi_{\text{CT}} = 2\pi M [n_r I_r(E, L) + n_\varphi L], \quad (4.2)$$

n_r and n_φ are *positive* co-primitive integers, M is a nonzero integer, ω_r is the radial frequency in Eq. (2.7). We took also into account explicitly that the actions \mathbf{I} (or $\{I_r, L\}$) are constants

of motion for the spherical integrable Hamiltonian, omitting in \mathbf{I} the upper indexes, related to their initial (prime) and final (double prime) values of Eq. (3.1). The integration limits in Eq. (4.1) for L are $-L_C \leq L \leq L_C$, where L_C is the maximum value corresponding to the circle orbit. All quantities in the integrand are taken at the energy surface $I_r = I_r(E, L)$ [Eq. (2.5)]. Thus, Eq. (4.1) is similar to the semiclassical Poisson summation trace formula which can be obtained directly by using the EBK quantization rules [6, 13] for the case of the spherical symmetry of the Hamiltonian. Note that formally, before taking the trace integral over the angular momentum L by the SPM in Eq. (4.1), one can consider positive and negative M , as those related to the two opposite directions of motion along a CT. They give, of course, equivalent contributions into the trace formula, due to a time-reversal symmetry of the Hamiltonian [or one may write simply the additional factor 2 in Eq. (4.1), but with a further summation over only positive integers M]. However, in contrast to the standard Poisson summation trace formula [13], there is no zero values of the integers in Eq. (4.1), $n_\varphi/n_r > 0$. In the derivations of Eq. (4.1) from Eq. (3.1), we essentially used that for *families* of the maximal degeneracy $\mathcal{K} = 1$, the generating function Φ_{CT} [Eq. (4.2)] becomes independent of the angle variables for the integrable Hamiltonian. Notice that in these derivations, the SPM conditions (3.4) and (3.5) were satisfied identically within the continuum of the stationary points $0 \leq \varphi, \Theta_r \leq 2\pi$, which form CTs, but they are not yet POs generally speaking for arbitrary angular momentum L . (Exceptions are the cases of the complete degeneracy like the spherical HO; see below.) The integration range in Eq. (4.1) taken from the minimum, $L_- = 0$, to the maximum, L_+ , value (for anticlockwise motion, for instance) covers the contributions of a whole manifold of closed and unclosed CTs of the tori in the phase space at the energy surface around the stationary point, $L = L^*$, which corresponds to the PO [23]. We shall specify the integration limits L_+ for the contribution of the ($\mathcal{K} = 1$) diameter families $M(n_r = 2, n_\varphi = 1)$ into Eq. (4.1) in Appendix D.

We apply then the stationary phase condition with respect to the variable L for the exponent phase Φ_{CT} [Eq. (4.2)] in the integrand of Eq. (4.1),

$$(\partial\Phi_{\text{CT}}/\partial L)^* = 0, \quad (4.3)$$

which is exactly the resonance condition (2.8). This condition determines the stationary phase point, $L = L^* = L_{\text{PO}}$, related to the POs $M(n_r, n_\varphi)$ of the families. Expanding now the exponent phase Φ_{CT} [Eq. (4.2)] in the variable L up to the second order, and assuming

that there is no singularities in the curvature (2.12) for the contribution of all $\mathcal{K} = 1$ families, one has

$$\Phi_{\text{CT}} = S_{\text{PO}}(E) + \frac{1}{2} J_{\text{PO}}^{(L)} (L - L^*)^2 + \dots, \quad (4.4)$$

where $S_{\text{PO}}(E)$ is the action along one of the isolated PO families, determined by Eq. (2.8),

$$S_{\text{PO}}(E) = 2\pi M [n_r I_r(E, L^*) + n_\varphi L^*]. \quad (4.5)$$

In this equation, M is the number of repetitions along the $PO = M(n_r, n_\varphi)$, $I_r(E, L)$ is the energy surface [Eq. (2.5)], $L = L^*(n_r, n_\varphi)$ is the solution of the PO equations (2.8) or (4.3). The Jacobian $J_{\text{PO}}^{(L)}$ in Eq. (4.4) measures the stability of the PO with respect to the variation of the angular momentum L at the energy surface,

$$J_{\text{PO}}^{(L)} = \left(\frac{\partial^2 S_{\text{CT}}}{\partial L^2} \right)_{L=L^*} = 2\pi M n_r K_{\text{PO}}, \quad (4.6)$$

$$K_{\text{PO}} = \left(\frac{\partial^2 I_r}{\partial L^2} \right)_{L=L_{\text{PO}}}, \quad (4.7)$$

where K_{PO} is the curvature (2.12), (B5) of the energy surface $I_r = I_r(E, L)$ at $L = L^* = L_{\text{PO}}$.

For the sake of simplicity, we shall discuss the simplest leading ISPM with the second order expansion (4.4) of the action phase, and with the zero order one of the pre-exponent factor over the L variable in $L - L^*$ in Eq. (4.1). Substituting now the expansion (4.4) into Eq. (4.1), one takes there the pre-exponential factor off the integral at $L = L^*$. Thus, applying Eq. (4.4), we are left with the integral over L of a Gaussian type integrand within the finite limits mentioned above for contributions of the one-parametric polygon-like and diameter families, including the contribution of boundaries for $0 < n_\varphi/n_r \leq 1/2$. Taking this integral over L within the finite limits, one obtains the ISPM trace formula, $\delta g^{(\mathcal{K})}(E)$, for contributions of the one-parametric ($\mathcal{K} = 1$) orbits,

$$\begin{aligned} \delta g^{(1)}(E) &= \text{Re} \sum_{\text{PO}} A_{\text{PO}}^{(1)}(E) \\ &\times \exp \left[\frac{i}{\hbar} S_{\text{PO}}(E) - i \frac{\pi}{2} \sigma_{\text{PO}} - i \phi_d \right]. \end{aligned} \quad (4.8)$$

The sum is taken over the discrete families of the PO $M(n_r, n_\varphi)$ with $n_r \geq 2n_\varphi$, $M \geq 1$ in the 2D RPL potential. $S_{\text{PO}}(E)$ is the action (4.5) along these POs. For the amplitudes $A_{\text{PO}}^{(1)}$ as for the $\mathcal{K} = 1$ families in the elliptic billiard [22], and the integrable Hénon-Heiles (IHH) potentials [23], one finds

$$A_{\text{PO}}^{(1)} = \frac{T_{\text{PO}}}{\pi \hbar^{3/2} \sqrt{M n_r^3 |K_{\text{PO}}|}} \text{erf}(\mathcal{Z}_{\text{PO}}^-, \mathcal{Z}_{\text{PO}}^+) \quad (4.9)$$

with the period

$$T_{\text{PO}} = \frac{2\pi n_r}{\omega_r} = \frac{2\pi n_\varphi}{\omega_\varphi} . \quad (4.10)$$

In Eq. (4.9), K_{PO} is the curvature of the energy surface $I_r = I_r(E, L)$ [see Eqs. (2.12), (B5)]. The generalized complex error function in Eq. (4.9) is introduced by

$$\text{erf}(u, v) = \frac{2}{\sqrt{\pi}} \int_u^v dz e^{-z^2} = \text{erf}(v) - \text{erf}(u) \quad (4.11)$$

with the standard error functions, $\text{erf}(z)$, of the complex arguments z . These arguments are specified by

$$\mathcal{Z}_{\text{PO}}^\pm = \sqrt{-i\pi M n_r |K_{\text{PO}}| / \hbar} (L_\pm - L_{\text{PO}}) , \quad (4.12)$$

$L_- = 0$ and $L_+ = L_C$ for all $\mathcal{K} = 1$ polygon-like PO families (besides of the diameters, see below). For simplicity, the finite integration interval of the angular momenta was split into two parts, $-L_C \leq L \leq 0$ and $0 \leq L \leq L_C$, where L_C is the angular momentum of a circle orbit, as mentioned above. There are the symmetric stationary points, $\pm|L^*|$, related to the anticlockwise and clockwise motions of the particle along the PO in two these phase-space parts. They give equivalent contributions to the amplitude, due to the independence of the Hamiltonian of time. Thus, we have reduced the integration region to $0 \leq L \leq L_C$, accounting for this time-reversibility symmetry simply by the factor 2 in Eq. (4.9) (exclusion is the diameters, for which one has one stationary point $L^* = 0$, and therefore, the time-reversibility degeneracy, taken into account automatically by the limits of the error functions, is one). For all the polygon-like and diameter POs ($n_r \geq 2$), we also found $L_- = 0$ for the minimum value of the angular momentum L .

For the Maslov index of the considered $\mathcal{K} = 1$ PO families and the constant phase ϕ_d in Eq. (4.8), one obtains

$$\sigma_{\text{PO}}^{(1)} = 2M n_r, \quad \phi_d = -\pi/4 . \quad (4.13)$$

The Maslov index σ_{PO} is determined in terms of the number of turning and caustic points by the Maslov&Fedoryuk catastrophe theory, see Refs. [4, 23–26]. Note that for the potentials with smooth edges, one has another expression for the Maslov index σ_{PO} than for the circular billiard [13, 27]. Note also that the total Maslov phase, defined as a sum of the asymptotic part (4.13) and the argument of the complex density amplitude (4.9), depends on the energy E and parameter α of the RPL potential [Eq. (1.1); see Refs. [11, 22]]. This total Maslov phase is changed through the bifurcation points smoothly, due to the phase of the complex error function in the amplitude (4.9) in Eq. (4.8).

For the stationary point L^* far from the ends of the physical integration interval, one can extend the integration range to the infinity from $-\infty$ to ∞ (in the case of diameters from zero to ∞). We then arrive asymptotically at the Berry&Tabor result [6] for the contribution of all $\mathcal{K} = 1$ families (4.8) with the following amplitude:

$$A_{\text{PO}}^{(1)} \rightarrow \frac{d_{\text{PO}} T_{\text{PO}}}{\pi \hbar^{3/2} \sqrt{M n_r^3 |K_{\text{PO}}|}} , \quad (4.14)$$

where d_{PO} takes into account the discrete degeneracy, $d_{\text{PO}} = 1$ for diameters $M(2, 1)$ ($n_r = 2n_\varphi$), and 2 for all other (polygon-like) POs ($n_r > 2n_\varphi$) [13]. In the circular billiard limit ($\alpha \rightarrow \infty$), the action $S_{\text{PO}}(E)$ is given by

$$S_{\text{PO}}(E) = p \mathcal{L}_{\text{PO}}, \quad p = \sqrt{2mE} , \quad (4.15)$$

with \mathcal{L}_{PO} being the length of the PO. For the curvature K_{PO} [see Eqs. (2.12) and (B5)], one can asymptotically ($\alpha \rightarrow \infty$) obtain

$$K_{\text{PO}} = \frac{1}{\pi p R_0 \sin(\pi n_\varphi / n_r)} . \quad (4.16)$$

Substituting the quantities (4.15), (4.16) and (4.13) into Eqs. (4.8) and (4.14), one obtains the well known trace formula for the circular billiard [13, 27]. Note that the amplitude (4.9) of the solution (4.8) is regular at the bifurcations which are the boundary points $L = L^* = L_C$ of the action (L) part of the tori as in the elliptic billiard [22].

Our SSPM result (4.14) coincides with the Berry and Tabor trace formula [6], as adopted to the 2D spherically-symmetric Hamiltonians by using the simplest expansions of the action phase and amplitude near the stationary point (see above), instead of a more general but more complicated mapping procedure; see more comments in Ref. [22]. The essential difference from the Berry&Tabor theory [6] is that Eq. (4.8) is one of the terms of solutions of the breaking-of-symmetry problem for the highest degenerate orbits, such as the one-parametric families in the IHH potential, or the elliptic and hyperbolic orbits in the elliptic billiard [22] (see also Refs. [12, 23]). Within the SPM of the extended Gutzwiller approach [5, 12, 23], we have to derive separately the contributions of the other orbits as the circle $\mathcal{K} = 0$ POs in the RPL potentials beyond the semiclassical Poisson summation-like trace formula (4.1) (with the restrictions to the range of the n_r and n_φ integer variables). Note that the ISPM trace formula (4.8) for the one-parametric families contains the end contributions related to the finite limits of integrations in the error functions. This trace formula is the contribution

of only such families, as essentially was used in its derivation from the trace formula (3.1) by taking 2π per the integral over the angle variable. Therefore, there is no contributions of the circle orbits in Eqs. (4.1) and (4.8). These orbits correspond to the *isolated stationary phase* point, but the derivation of this section IV A cannot be applied for their contributions within the ISPM. This is similar to that for the IHH potential [23].

Notice also that all roots $L = L^* = L_{\text{PO}}$ of the stationary phase equation (2.8) for $\mathcal{K} = 1$ families $PO = M(n_r, n_\varphi)$ are in between the minimum value $L = L^* = 0$ for diameters, and a maximum one $L = L_C$, $0 \leq L_{\text{PO}} \leq L_C$ (anticlockwise motion, for example). There is the additional isolated stationary-boundary point $L^* = L_C$, which gives the separate contribution of the circle orbits into the phase space trace formula, see below.

B. Circle orbits ($\mathcal{K} = 0$)

In contrast to the derivations of contributions of the most degenerated ($\mathcal{K} = 1$) orbits, we take now into account existence of the isolated stationary point of the phase Φ_{CT} (3.2) in the radial spherical phase-space variables $r'^* = r''^* = r_C$, $p_r'^* = p_r''^* = 0$ [Eqs. (3.4) and (3.5)]. After the transformation of the integration variables in Eq. (3.1) to the spherical phase space coordinates $\{r', \varphi'; p_r'', L\}$, it is convenient first to perform the exact integrations over L by using the energy conserving δ -function, and over the cycle azimuthal angle φ' leading simply to 2π as above ($\int d\varphi'/\omega_\varphi = T_{\varphi, \text{CT}}$ is the primitive rotation period). Thus, as in the subsection A, one finds

$$g_{\text{scl}}(E) = \frac{1}{(2\pi\hbar)^2} \text{Re} \sum_{\text{CT}} \int dr' \int dp_r'' T_{\varphi, \text{CT}} \times |\mathcal{J}_{\text{CT}}(p_r', p_r'')|^{1/2} \exp \left[\frac{i}{\hbar} \Phi_{\text{CT}} - i \frac{\pi}{2} \mu_{\text{CT}} \right]. \quad (4.17)$$

The stationary phase condition for the SPM integration over the radial momentum p_r'' in Eq. (4.17) writes

$$\left(\frac{\partial \Phi_{\text{CT}}}{\partial p_r''} \right)^* \equiv (r' - r'')^* = 0. \quad (4.18)$$

The solution of this equation is the isolated stationary point $p_r'' = p_r''^* = p_r^* = 0$. The phase Φ_{CT} [Eq. (3.2)] is expanded in the momentum p_r'' near this point $p_r''^* = 0$ in power series,

$$\Phi_{\text{CT}} = \Phi_{\text{CT}}^* + \frac{1}{2} \mathcal{J}_{\text{CT}}^{(p)} (p_r'' - p_r^*)^2 + \cdots, \quad (4.19)$$

where the Jacobian $\mathcal{J}_{\text{CT}}^{(p)}$ is given by

$$\mathcal{J}_{\text{CT}}^{(p)} = \left(\frac{\partial^2 \Phi_{\text{CT}}}{\partial p_r''^2} \right)^* = \left[\frac{2\pi M n_r K}{(\partial p_r'' / \partial L)^2} \right]^* . \quad (4.20)$$

The upper index star means again that the corresponding quantity is taken at the stationary point, $p_r'' = p_r''^* = 0$. By using the 2nd order expansion of the exponent phase (4.19) and taking the pre-exponent amplitude factor off the integral at this stationary point, one gets the internal integral over p_r'' in terms of the error function as in the previous section. According to Eq. (3.2), with the radial-coordinate closing condition (4.18) for the CTs, the short phase Φ_{CT}^* in Eq. (4.19) can be written in terms of the corresponding variables as $\Phi_{\text{CT}}^* = \int_{r'}^{r''} p_r \, dr$. Taking then into account the CT closing condition (4.18), $r' = r'' = r$, for the stationary phase equation in the integration over the radial r coordinate perpendicular to the circle orbit, one results in

$$\left(\frac{\partial \Phi_{\text{CT}}^*}{\partial r''} + \frac{\partial \Phi_{\text{CT}}^*}{\partial r'} \right)^* \equiv (p_r'' - p_r')^* = 0 . \quad (4.21)$$

Therefore, together with Eq. (4.18), one has the PO conditions related to the circular orbit $r = r^* = r_C$ and $L = L^* = L_C$ (see Appendix A). As usually within the SPM, we expand now the phase Φ_{CT}^* in the radial coordinate r near this $r^* = r_C$,

$$\Phi_{\text{CT}}^* = M S_C + \frac{1}{2} \mathcal{J}_{MC}^{(r)} (r - r_C)^2 + \dots , \quad (4.22)$$

where S_C is the action along the primitive circle PO (C),

$$\begin{aligned} \mathcal{J}_{MC}^{(r)} &= \left(\frac{\partial^2 S_{\text{CT}}}{\partial r'^2} + 2 \frac{\partial^2 S_{\text{CT}}}{\partial r' \partial r''} + \frac{\partial^2 S_{\text{CT}}}{\partial r''^2} \right)_{MC}^* \\ &= \left(-\frac{\partial p_r'}{\partial r'} - 2 \frac{\partial p_r'}{\partial r''} + \frac{\partial p_r''}{\partial r''} \right)_{MC}^* . \end{aligned} \quad (4.23)$$

Using now the action phase expansion (4.22) at the second order as the simplest ISPM approximation, and taking the pre-exponent amplitude factor at the isolated stationary point $r = r_C$ off the integral, one finally obtains

$$\begin{aligned} \delta g_{\{MC\}}^{(0)}(E) &= \text{Re} \sum_{M=1}^{\infty} A_{MC}^{(0)}(E) \\ &\times \exp \left[\frac{i}{\hbar} M S_C(E) - i \frac{\pi}{2} \sigma_{MC}^{(0)} - i \phi_d^{(0)} \right] . \end{aligned} \quad (4.24)$$

The sum runs all circle orbits MC , $M = 1, 2, \dots$ are positive integers. The action $S_C(E)$ along the primitive C orbit is given by

$$S_C(E) = \oint_C p_\varphi \, d\varphi = 2\pi L_C \quad (4.25)$$

with L_C , given explicitly in Eq. (A2). In Eq. (4.24), $\sigma_{MC}^{(0)}$ is the Maslov index determined by the number of caustic and turning points along the circle orbit, according to the Fedoryuk& Maslov catastrophe theory (see Refs. [23–26]),

$$\sigma_{MC}^{(0)} = 4M, \quad \phi_d^{(0)} = 0. \quad (4.26)$$

For the amplitudes $A_{MC}^{(0)}(E)$ in Eq. (4.24), one finds

$$A_{MC}^{(0)} = \frac{T_C}{2\pi\hbar\sqrt{F_{MC}}} \operatorname{erf}(\mathcal{Z}_{p,MC}^{(-)}, \mathcal{Z}_{p,MC}^{(+)}) \operatorname{erf}(\mathcal{Z}_{r,MC}^{(-)}, \mathcal{Z}_{r,MC}^{(+)}) , \quad (4.27)$$

where T_C is the period of the primitive ($M = 1$) orbit C ,

$$T_C = 2\pi m r_C^2 / L_C , \quad (4.28)$$

see Eq. (A2) for r_C and L_C . In Eq. (4.27), F_{MC} is the Gutzwiller stability factor [3] of the circle orbits [Eq. (2.9)]. The arguments of the error functions in Eq. (4.27) can be transformed to the following invariant form (see Appendix E):

$$\begin{aligned} \mathcal{Z}_{p,MC}^{(\pm)} &= \sqrt{\frac{i}{\hbar} \pi M \sqrt{\alpha+2} |K_C|} (L_{\pm} - L_C) , \\ &\quad L_+ = L_C, \quad L_- = 0 , \\ \mathcal{Z}_{r,MC}^{(\pm)} &= \sqrt{-\frac{i F_{MC}}{4\pi M \hbar (\alpha+2)^{3/2} |K_C|}} \Theta_r^{(\pm)} , \\ &\quad \Theta_r^{(+)} = 2\pi, \quad \Theta_r^{(-)} = 0 . \end{aligned} \quad (4.29)$$

Here, L_{\pm} are the maximum and minimum values of the angular-momentum integration variable for the contribution of the circle orbits, K_C is their curvature (see Appendix E),

$$K_C = -\frac{(\alpha+1)(\alpha-2)}{12(\sqrt{\alpha+2})^3 L_C} . \quad (4.30)$$

For $\alpha = 4$; the period T_C [Eq. (4.28)], action S_C [Eq. (4.25)], curvature K_C [Eq. (4.30)], and stability factor F_{MC} [Eq. (2.9)] for the circle orbits are identical to those obtained in Ref. [18]. We used also the properties of the Jacobians for transformations of the different coordinates, in particular, given by Eq. (E3). The time-reversibility of the Hamiltonian was similarly taken into account in Eq. (4.24), as explained above. Note that after applying the stationary phase conditions $r^* = r_C$ [Eq. 4.18] and $p_r^* = 0$ [Eq. (4.21)], the angular momentum L of the circular orbits as function of the r and p_r becomes the stationary point $L^* = L_C$

at the boundary of the classically accessible phase space. Notice also that the asymptotic Maslov phase is defined traditionally in terms of the Maslov index $\sigma_{MC}^{(0)}$ [Eq. (4.13)]. There is again the two components of the Maslov phase in the ISPM trace formula (4.24) for the MC orbits. One of them is the asymptotic constant part (4.13) independent of the energy. Another part is the argument of the complex amplitudes $A_{MC}^{(0)}$ [Eq. (4.27)], that changes continuously through the bifurcation points. The total Maslov phase for the circle POs is given by the sum of these two contributions. It ensures a smooth transition of the trace formula (4.24) for the contribution of the circle POs through the bifurcation points.

In the asymptotic limit of the finite non-zero integration boundaries, $L_- \rightarrow -\infty$ and $\Theta_r^+ \rightarrow \infty$, i.e., far from any bifurcations α_{bif} (also from the HO symmetry breaking at $\alpha = 2$), the expression (4.27) tends to the amplitude of the Gutzwiller trace formula for isolated orbits [3, 13]. The simplest approximation is $L_+ = L_C$, $L_- = 0$; and $\Theta_r^- = 0$, $\Theta_r^{(+)} = 2\pi$, which correspond to the total physical phase space accessible for the classical motion. The factors $\sqrt{\alpha + 2}$ in front of the curvature K_C in Eq. (4.29) appear because of the frequency ratio Eq. (2.8) for ω_φ/ω_r for the circle orbits for any parameter $\alpha \geq 2$. Taking Eq. (4.27) asymptotically far from the bifurcations α_{bif} [Eq. (2.11)], one has the amplitude of the Gutzwiller trace formula:

$$A_{MC}^{(0)}(E) \rightarrow \frac{1}{2\pi \hbar} \frac{T_C}{\sqrt{F_{MC}}} . \quad (4.31)$$

In this limit, the asymptotic Maslov index $\sigma_{MC}^{(0)}$ and $\phi_d^{(0)}$ in Eq. (4.8) are given by Eq. (4.26).

For the opposite limit to the bifurcations ($F_{MC} \rightarrow 0$, when $\alpha \rightarrow \alpha_{\text{bif}}$), one finds that the both arguments of the second error function in Eq. (4.27) tend to zero as $\sqrt{|F_{MC}|}$, see Eq. (4.29). The Gutzwiller stability factor F_{MC} , going to zero, is exactly canceled by the same one in the denominator, and we arrive at

$$A_{MC}^{(0)}(E) \rightarrow \frac{T_C}{\pi \hbar^{3/2} \sqrt{M(\alpha + 2)^{3/2} |K_C|}} \times \text{erf} \left(\mathcal{Z}_{p, MC}^{(-)}, \mathcal{Z}_{p, MC}^{(+)} \right) e^{-i\pi/4} . \quad (4.32)$$

Thus, in contrast to the Gutzwiller SSPM, one obtains the finite result at the bifurcations within the ISPM. Notice that the enhancement in order of $\hbar^{-1/2}$ with respect to the Gutzwiller asymptotic amplitude (4.31) takes place locally near the bifurcation points. Note also that at the circular billiard limit, when $K_C \rightarrow \infty$ (separatrix), one finds a continuous limit which is zero in the case of the PRL potential.

C. Total trace formula for the oscillating level density

The total semiclassical oscillating (shell) correction to the level density (3.1) for the RPL potentials in two dimensions is thus given by

$$\delta g_{\text{scl}}(E) = \delta g_{\text{scl}}^{(1)}(E) + \delta g_{\text{scl}}^{(0)}(E) , \quad (4.33)$$

where

$$\begin{aligned} \delta g_{\text{scl}}^{(\mathcal{K})}(E) &= \text{Re} \sum_{\text{PO}} A_{\text{PO}}^{(\mathcal{K})}(E) \\ &\times \exp \left[\frac{i}{\hbar} S_{\text{PO}}(E) - i \frac{\pi}{2} \sigma_{\text{PO}}^{(\mathcal{K})} - i \phi_d^{(\mathcal{K})} \right] . \end{aligned} \quad (4.34)$$

The amplitudes $A_{\text{PO}}^{(\mathcal{K})}$ [see Eqs. (4.9) for $\mathcal{K} = 1$ and (4.27) for $\mathcal{K} = 0$], actions S_{PO} , Maslov indexes $\sigma_{\text{PO}}^{(\mathcal{K})}$, and constant phases $\phi_d^{(\mathcal{K})}$ [Eqs. (4.13) and (4.26)] were specified in the two previous sections IV A and IV B, respectively.

Using the scale invariance (2.2), one may factorize the action integral $S_{\text{PO}}(E)$,

$$\begin{aligned} S_{\text{PO}}(E) &= \oint_{\text{PO}(E)} \mathbf{p} \cdot d\mathbf{r} = \left(\frac{E}{E_0} \right)^{\frac{1}{2} + \frac{1}{\alpha}} \oint_{\text{PO}(E=E_0)} \mathbf{p} \cdot d\mathbf{r} \\ &\equiv \varepsilon \tau_{\text{PO}} . \end{aligned} \quad (4.35)$$

In the last equation, we define the scaled energy ε and scaled period τ_{PO} by

$$\varepsilon = \left(\frac{E}{E_0} \right)^{\frac{1}{2} + \frac{1}{\alpha}} , \quad \tau_{\text{PO}} = \oint_{\text{PO}(E=E_0)} \mathbf{p} \cdot d\mathbf{r} . \quad (4.36)$$

To realize the advantage of the scaling invariance (2.2), it is helpful to use the scaled energy (period) in place of the corresponding original variables. For the HO one has $\alpha = 2$, and the scaled energy and period are proportional to the unscaled quantities. For the cavity potential ($\alpha \rightarrow \infty$), they are proportional to the momentum p and length \mathcal{L}_{PO} , respectively.

Using the transformation of the energy E to the scaled energy ε , one can introduce the dimensionless scaled-energy level density. The advantage of this transformation is that a nice plateau condition is always found in the Strutinsky SCM smoothing procedure by using the scaled spectrum ε_i (see Refs. [11, 22] for the case of the billiard limit $\alpha \rightarrow \infty$). Then, one can use a simple relation between the original and scaled-energy level densities,

$$\mathcal{G}(\varepsilon) = \sum_i \delta(\varepsilon - \varepsilon_i) = g(E) \frac{dE}{d\varepsilon} . \quad (4.37)$$

For the corresponding semiclassical formula for the oscillating part of the level density, one finds

$$\begin{aligned}
\delta\mathcal{G}_{\text{scl}}^{(\mathcal{K})}(\varepsilon) &= \frac{dE}{d\varepsilon} \delta g^{(\mathcal{K})}(E) = \sum_{\text{PO}} \delta\mathcal{G}_{\text{PO}}^{(\mathcal{K})}(\varepsilon) \\
&= \text{Re} \sum_{\text{PO}} \mathcal{A}_{\text{PO}}^{(\mathcal{K})}(\varepsilon) \exp \left[\frac{i}{\hbar} \varepsilon \tau_{\text{PO}} - \frac{i\pi}{2} \sigma_{\text{PO}}^{(\mathcal{K})} - i\phi_d^{(\mathcal{K})} \right], \\
\mathcal{A}_{\text{PO}}^{(\mathcal{K})}(\varepsilon) &= \frac{dE}{d\varepsilon} A_{\text{PO}}^{(\mathcal{K})}(E).
\end{aligned} \tag{4.38}$$

The simple form of the phase function (4.35) enables us also to make easy use of the Fourier transformation technique. The Fourier transform of the semiclassical scaled-energy level density with respect to the scaled period τ is given by

$$F(\tau) = \int d\varepsilon \mathcal{G}(\varepsilon) e^{i\varepsilon\tau/\hbar} \approx F_0(\tau) + \sum_{\text{PO}} \tilde{\mathcal{A}}_{\text{PO}} \delta(\tau - \tau_{\text{PO}}), \tag{4.39}$$

which exhibits peaks at periodic orbits $\tau = \tau_{\text{PO}}$. $F_0(\tau)$ represents the Fourier transform of the smooth Thomas-Fermi (TF) level density and has a peak at $\tau = 0$ related to the zero-action trajectory [12]. Thus, from the Fourier transform of the scaled-energy quantum-mechanical level density,

$$\begin{aligned}
F(\tau) &= \int \left[\sum_i \delta(\varepsilon - \varepsilon_i) \right] e^{i\varepsilon\tau/\hbar} d\varepsilon = \sum_i e^{i\varepsilon_i\tau/\hbar}, \\
\varepsilon_i &= \left(\frac{E_i}{E_0} \right)^{\frac{1}{2} + \frac{1}{\alpha}},
\end{aligned} \tag{4.40}$$

one can directly extract the information about classical PO contributions. The trace formula (4.33) has the correct asymptotic SSPM limits to the Berry and Tabor results (4.8), (4.14) for $\mathcal{K} = 1$ polygon-like (including the diameters) and to the Gutzwiller trace formula (4.24), (4.31) for $\mathcal{K} = 0$ circle POs. As shown in the sections IV A and IV B, one obtains also the limit of the trace formula [Eqs. (4.33) and (4.34)] to that of the circular billiard $\alpha \rightarrow \infty$ [13, 27]. In this limit one has obviously zero for the circular-orbit contributions as for the potential barrier separatrix in the IHH potential [23].

D. Harmonic oscillator limit

In the isotropic harmonic oscillator limit [$\alpha \rightarrow 2$ in the power-law potential (1.1)], the energy surface is simplified to the linear function in actions,

$$E = \omega_r I_r + \omega_\varphi I_\varphi = \omega_\varphi (2 I_r + L). \tag{4.41}$$

Therefore, in this limit the curvature K_{PO} for all POs [including the maximum value $L = L_C = E/\omega_\varphi$ for the circle orbits, Eq. (4.30), and $L = 0$ for diameter ones, Eq. (D3)] and stability factor F_{MC} [Eq. (2.9)] for the MC orbits turn into zero. However, there is no singularities in the ISPM trace formulas (4.8) for the contributions of all $\mathcal{K} = 1$ families and (4.24) for the circle orbits in the limits $K_{\text{PO}} \rightarrow 0$ and $F_{MC} \rightarrow 0$. The arguments of both error functions, $\propto \sqrt{K_C}$ and $\propto \sqrt{F_{MC}/K_C}$ in Eq. (4.27), for instance, approach zero and singularities are canceled with the same ones in the denominators of the multipliers in front of them, and similarly, in Eq. (4.8) for one error function; see Eqs. (4.9), (4.12), (4.27) and (4.29) with the help of Eq. (4.32). Therefore, one has a continuous limit of the total trace formula (4.33) for $\alpha \rightarrow 2$. Moreover, in this limit, one obtains exactly the same half of the HO trace formula for the MC orbit contribution (4.24) and the $M(2, 1)$ diameter one [Eq. (4.8)] up to the relatively small higher-order corrections in \hbar [see also Eq. (3.68) of Sec. 3.2.4 in Ref. [13]],

$$g_{\{MC\}}^{(0)}(E) \rightarrow \frac{1}{2} \delta g_{\text{HO}}^{(2)}(E), \quad g_{\{MD\}}^{(1)}(E) \rightarrow \frac{1}{2} \delta g_{\text{HO}}^{(2)}(E). \quad (4.42)$$

Here, $\{MC\}$ and $\{MD\}$ represent sum of all repetitions of circle and diameter orbits, $M = 1, 2, \dots$, respectively. Thus, the HO limit of the sum of the circle and diameter orbit contributions is exactly analytically given by the HO trace formula. We point out that for $\alpha \rightarrow 2$, the contributions of circle MC and diameter $M(2, 1)$ orbits encounter local increases of the degeneracies \mathcal{K} by 2 and 1 units, respectively.

As noted above, in the HO limit $\alpha \rightarrow 2$, only the diameter $M(2, 1)$ and the circle MC (both with repetitions) survive, and they form $\mathcal{K} = 2$ families in the HO potential. Taking into account also that the angular momentum for the diameters is always zero, $L^* = 0$, and for the circle orbits $L^* = L_C$, we shall assume that the integration over L for the diameters is performed from $L_- = 0$ to $L_+ = L_C/2$ and for the circle orbits from $L_- = 0$ to $L_+ = L_C$, such that they give naturally equivalent contributions into the HO trace formula, as shown in Eq. (4.42), see also Ref. [23]. The difference is in the integration limits for the circle orbits [see Eq. (4.29)], in contrast to Eqs. (4.12) and (D2) for the diameter boundaries. Notice that the contribution of the polygon-like one-parametric orbits, $\delta g^{(1)}(E)$, disappears in the ghost HO limit. Thus, one obtains the continuous transition of the oscillating part of the ISPM level density $\delta g_{\text{scl}}(E)$ through all bifurcation points, including the HO symmetry breaking.

E. Averaged level density

For comparison with the quantum level densities obtained by the SCM, we need also to average the trace formula (4.33) over the spectrum. As this trace formula is given by the sum of the separating periodic-orbit terms everywhere (including the bifurcations), one can approximately take the integral of the local averaging over energies (folding integral) analytically in terms of the Gaussian weight factor at the Gaussian width parameter $\Gamma \ll E_F$. As result, for the averaged density $\delta g_r(E)$ with this weight function, one obtains (see Refs. [5, 12, 13])

$$\delta g_r(E) = \sum_{\text{PO}} \delta g_{\text{PO}}(E) \exp \left[- (t_{\text{PO}} \Gamma / \hbar)^2 \right] . \quad (4.43)$$

The total level density is calculated analytically by adding the TF smooth component $g_{\text{TF}}(E)$ [13],

$$g_{\text{T}}(E) = g_{\text{TF}}(E) + \delta g_{\text{T}}(E), \quad (4.44)$$

where

$$\begin{aligned} g_{\text{TF}}(E) &= \frac{1}{(2\pi\hbar)^2} \int d\mathbf{r} \int d\mathbf{p} \delta(E - H(p, r)) \\ &= \frac{mr_{\text{max}}^2}{2\hbar^2} = \frac{1}{2E_0} \left(\frac{E}{E_0} \right)^{2/\alpha} . \end{aligned} \quad (4.45)$$

Here, $H(p, r) = p^2/2m + V(r)$, $p(r)$ is given by Eq. (2.4), and r_{max} is the turning point as one of solutions of the equation $V(r) = E$. For the RPL potential (2.1), $r_{\text{max}} = R_0(E/E_0)^{1/\alpha}$, and we used the relation $E_0 = \hbar^2/mR_0^2$ in the last equation of (4.45).

For the averaged scaled-energy level density, one obtains the semiclassical trace formula:

$$\begin{aligned} \delta \mathcal{G}_\gamma(\varepsilon) &= \sum_{\kappa=0}^1 \delta \mathcal{G}_\gamma^{(\kappa)}(\varepsilon) \\ &= \sum_{\kappa=0}^1 \sum_{\text{PO}} \delta \mathcal{G}_{\text{PO}}^{(\kappa)}(\varepsilon) \exp \left[- \left(\frac{\tau_{\text{PO}} \gamma}{2\hbar} \right)^2 \right] , \end{aligned} \quad (4.46)$$

where $\delta \mathcal{G}_{\text{PO}}^{(\kappa)}(\varepsilon)$ is given by Eq. (4.38), γ is the dimensionless Gaussian width parameter used for the coarse-graining over the scaled spectrum ε_i . For the Thomas-Fermi density, one obtains the simple expression:

$$\mathcal{G}_{\text{TF}}(\varepsilon) = g_{\text{TF}}(E) \frac{dE}{d\varepsilon} = \frac{\alpha}{2 + \alpha} \varepsilon . \quad (4.47)$$

F. The shell correction energies

The PO expansion for the semiclassical shell correction energies δU_{scl} is given by [5, 11–13, 22]

$$\delta U_{\text{scl}} = 2 \sum_{\text{PO}} \frac{\hbar^2}{t_{\text{PO}}^2} \delta g_{\text{PO}}(E_F), \quad (4.48)$$

where $t_{\text{CT}} = MT_{\text{PO}}(E_F)$ is the period of particle motion along the PO (taking into account its repetition number M) at the Fermi energy $E = E_F$. The Fermi energy E_F as function of the particle number N is determined by the particle number conservation,

$$N = 2 \sum_i n_i = 2 \int_0^{E_F} dE g(E), \quad (4.49)$$

where $n_i = \theta(E_F - E_i)$ are the occupation numbers. The factors 2 in Eqs. (4.48) and (4.49) account for the spin degeneracy of Fermi particles with spin 1/2.

Note that the shell correction energies δU which are the observed physical quantities do not contain an arbitrary averaging parameter Γ , in contrast to the level density $g_r(E)$. The convergence of the PO sum (4.48) is ensured by the additional factor in front of the oscillating density components δg_{PO} which is inversely proportional to square of the PO period t_{PO} . This means that we need shorter orbits if they occupy enough large phase-space volume.

In the quantum SCM calculations, the shell correction energies are usually obtained by extracting the oscillating part from a sum of the single-particle energies,

$$U = 2 \sum_i n_i E_i. \quad (4.50)$$

However, the direct application of the SCM average procedure to the spectra E_i of RPL potentials (except for the HO limit) does not give any good plateau condition as for the level density $g(E)$ in Eq. (4.37). On the other hand, there is no problems in calculations of the shell-correction energies because one can always find rather a good plateau in the SCM application to a helpful sum of the single-particle scaled energies ε_i ,

$$\mathcal{U} = 2 \sum_i n_i \varepsilon_i, \quad (4.51)$$

which is simply related to the original shell correction energy. Indeed, applying exactly the same derivations of Eq. (4.48) to the semiclassical trace formula for the oscillating part of

Eq. (4.51), one gets

$$\delta\mathcal{U}_{\text{scl}} = 2 \sum_{\text{PO}} \frac{\hbar^2}{\tau_{\text{PO}}^2} \delta\mathcal{G}_{\text{PO}}(\varepsilon_F). \quad (4.52)$$

Here, the scaled Fermi energy ε_F is determined by

$$N = 2 \int_0^{\varepsilon_F} \mathcal{G}(\varepsilon) d\varepsilon. \quad (4.53)$$

Using now the obvious relations

$$t_{\text{PO}} = \frac{\tau_{\text{PO}}}{dE/d\varepsilon}, \quad \delta g_{\text{PO}}(E) = \frac{\delta\mathcal{G}_{\text{PO}}(\varepsilon)}{dE/d\varepsilon}$$

in Eq. (4.48) one obtains

$$\delta U_{\text{scl}} = \left(\frac{dE}{d\varepsilon} \right)_{\varepsilon_F} \delta\mathcal{U}_{\text{scl}}. \quad (4.54)$$

Thus, we arrive at the simple relation between the original shell-correction energy δU [Eq. (4.48)] and the scaled one $\delta\mathcal{U}$, valid for both semiclassical and quantum (neglecting the second order terms in the shell fluctuations of the Fermi energy) calculations:

$$\delta U = \left(\frac{dE}{d\varepsilon} \right)_{\varepsilon_F} \delta\mathcal{U} = E_0 \frac{2\alpha}{\alpha+2} \varepsilon_F^{(\alpha-2)/(\alpha+2)} \delta\mathcal{U}. \quad (4.55)$$

This relation can be also directly obtained by using the standard quantum SCM relations of the first-order shell-correction energy δU to the oscillating part of the level density $\delta g(E)$ up to the same second order terms in the Fermi energy oscillations [2],

$$\delta U = 2 \sum_i \delta n_i E_i = 2 \int_0^{E_F} dE (E - E_F) \delta g(E), \quad (4.56)$$

and corresponding ones for the scaled quantities,

$$\delta\mathcal{U} = 2 \sum_i \delta n_i \varepsilon_i = 2 \int_0^{\varepsilon_F} d\varepsilon (\varepsilon - \varepsilon_F) \delta\mathcal{G}(\varepsilon). \quad (4.57)$$

In these derivations, $\delta n_i = n_i - \tilde{n}_i$, \tilde{n}_i are the smooth occupation numbers of the SCM. We applied also the usual transformations from the Fermi energies to the particle numbers by using Eqs. (4.49) and (4.53), as well as the definitions of the averaged Fermi energy \tilde{E}_F , and the scaled one $\tilde{\varepsilon}_F$,

$$N = 2 \int_0^{\tilde{E}_F} dE \tilde{g}(E) = 2 \int_0^{\tilde{\varepsilon}_F} d\varepsilon \tilde{\mathcal{G}}(\varepsilon). \quad (4.58)$$

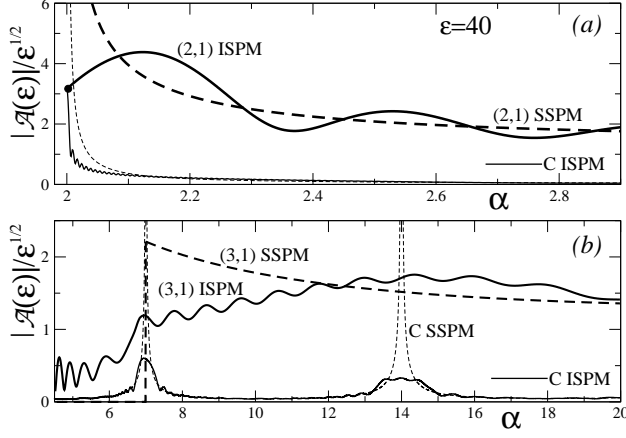


FIG. 2. Moduli of the scaled amplitudes $|\mathcal{A}_{\text{PO}}(\varepsilon)|$ as functions of α for the primitive ($M = 1$) circle C [see Eq. (4.27)], diameter (2,1) and triangle-like (3,1) [see Eq. (4.9)] POs, in units of $\varepsilon^{1/2}$ at the scaled energy $\varepsilon = 40$. The panel (a) shows the HO limits ($\alpha \rightarrow 2$) of the improved stationary-phase method (ISPM, solid), and standard stationary-phase method (SSPM, dashed curve) amplitudes for the circle C (thin) and diameter (2,1) (thick curve) orbits. The filled circle denotes one half of the HO amplitude (4.42) at $\alpha = 2$. In the panel (b), ISPM (solid) and SSPM (dashed curve) amplitudes for the circle C (thin) and triangle-like (3,1) (thick curve) POs are shown.

V. AMPLITUDE ENHANCEMENT AND COMPARISON WITH QUANTUM RESULTS

A remarkable enhancement of the ISPM amplitudes in PO sums for the oscillating level density (4.34) and shell correction energy (4.48) is the general characteristic property of the periodic orbits, due to their bifurcation (symmetry-breaking) scenarios where the newborn POs emerge from the existing parent ones. In Fig. 2, the scaled amplitudes $|\mathcal{A}_{\text{PO}}|$, divided by $\varepsilon^{1/2}$ to normalize the energy dependence for $\mathcal{K} = 1$ orbits, are presented for several shortest POs as functions of the power parameter α in order to show the typical bifurcation enhancement phenomena. In Fig. 2(a), the enhancement of the primitive diameter (2,1) amplitudes $|\mathcal{A}_{(2,1)}|$ [Eq. (4.9)], and those $|\mathcal{A}_{MC}|$ [Eq. (4.27)] for the primitive circle orbit C are clearly seen in the HO limit $\alpha \rightarrow 2$; see also Eq. (4.38). Figure 2(b) shows the enhancement of the shortest orbit C around the bifurcation point $\alpha = 7$, and the birth of the triangle-like orbit (3,1) there. Note that the ISPM amplitude $|\mathcal{A}_{(3,1)}|$ [Eq. (4.9)] of the (3,1) orbit keeps its magnitude up to rather a large value of α above the bifurcation

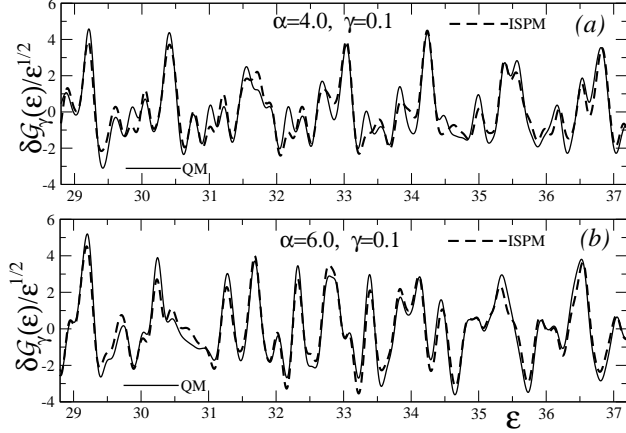


FIG. 3. The oscillating part of the level density $\delta\mathcal{G}_\gamma(\varepsilon)$ in units of $\varepsilon^{1/2}$ vs the scaled energy ε for $\alpha = 4.0$ (a), and $\alpha = 6.0$ (b), at the (dimensionless) width parameter $\gamma = 0.1$ in the Gaussian coarse-graining over the scaled energies: QM (solid curve) is the quantum-mechanical results (using the Strutinsky SCM with the averaging width $\tilde{\gamma} \approx 3$ and curvature correction polynomial of the 6th power); ISPM (dashed curve) is the semiclassical results using the improved stationary phase method.

($\alpha > \alpha_{\text{bif}}$). The ISPM amplitude for the circle orbit C exhibits a remarkable enhancement at the bifurcation point $\alpha = 7$. The divergence of its SSPM amplitude at the bifurcation point is successfully removed. As also seen from Fig. 2(b), the ISPM amplitude for the (3, 1) PO is continuously changed through this bifurcation, in contrast to the discontinuity of the SSPM amplitude. This orbit exists, in fact, only at $\alpha \geq 7$, and the amplitude in the region $\alpha < 7$ is due to the formal stationary point which has no direct sense in the classical dynamics. Therefore, the corresponding PO is called usually as a ghost orbit [11, 13, 23]. An oscillatory behavior of the amplitude $|\mathcal{A}_{(3,1)}|$ in the ghost region far from the bifurcation has no physical significance, since it is washed out in the averaged coarse-graining level density by an rapidly oscillating phase of the complex amplitude $\mathcal{A}_{(3,1)}$; see Refs. [11, 23]. These ghost amplitude oscillations are suppressed even more by using higher order expansions in the phase and amplitudes in a more precise ISPM (Ref. [11]).

Figures 3–7 show the oscillating part of the semiclassical scaled-energy level density $\delta\mathcal{G}_\gamma(\varepsilon)$ [Eq. (4.46)] in units of $\varepsilon^{1/2}$ as functions of the scaled energy ε for several values of the power parameter α and the averaging width γ . The ISPM semiclassical results show good agreement with the quantum mechanical (QM) ones for a transition from the gross to fine

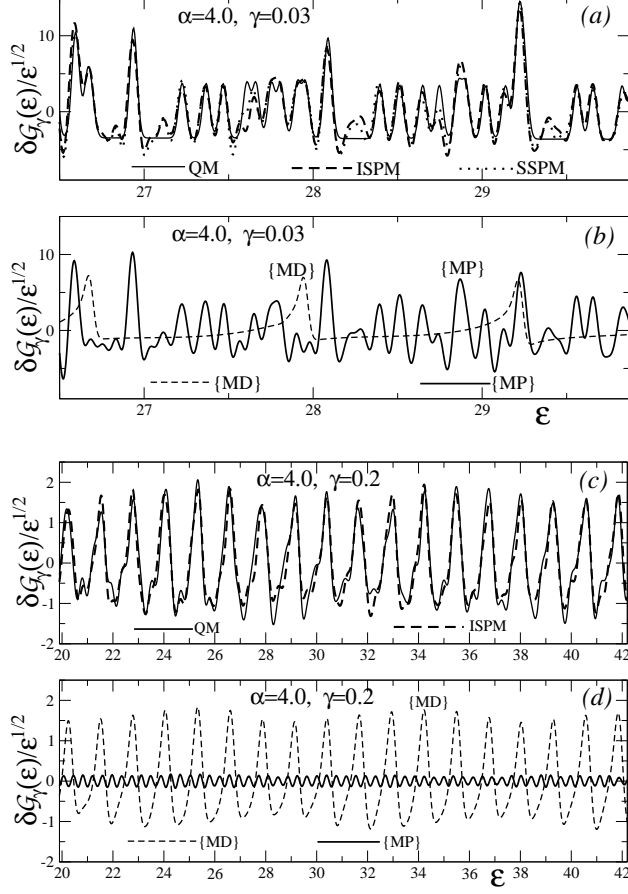


FIG. 4. The same as in Fig. 3 for $\alpha = 4.0$ but with other width parameters, $\gamma = 0.03$ (a,b) and $\gamma = 0.2$ (c,d). *Panels (a,c)*: QM (solid line) is the quantum-mechanical results; ISPM (dashed) is the semiclassical results using the improved stationary phase method; SSPM (dotted) is the results of the standard stationary phase method [shown only in the panel (a)]. *Panels (b,d)*: {MD} (dashed) is the contribution of the diameters (including their repetitions) and {MP} (thin solid) for other $\mathcal{K} = 1$ polygon-like POs.

resolutions of the spectra. The QM calculations are carried out by the use of the standard Strutinsky averaging over the scaled energy ϵ , in which we find a good plateau around the Gaussian averaging width $\tilde{\gamma} = 2 - 3$ with the even curvature correction polynomials of 4th to 8th powers.

For the powers $\alpha = 4.0$ and 6.0 , one finds a good agreement with the SSPM asymptotic behavior [see Figs. 4(a) and 5(a)] because they are sufficiently far from the bifurcation points $\alpha = 4.25$ and 7.0 which correspond to the birth of the star-like $(5, 2)$ and triangle-like $(3, 1)$

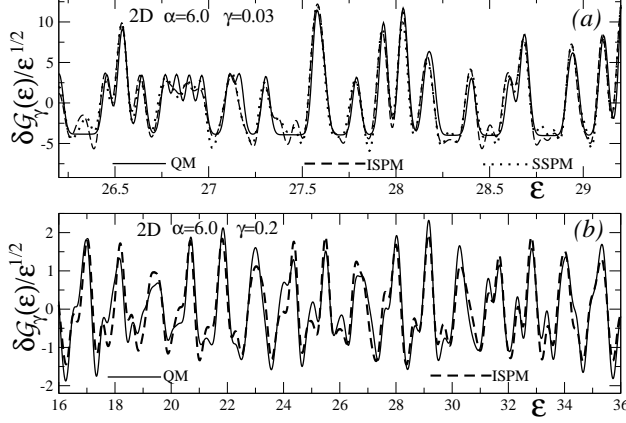


FIG. 5. The same as in Fig. 3 for $\alpha = 6.0$, but with other width parameters, $\gamma = 0.03$ (a), and $\gamma = 0.2$ (b).

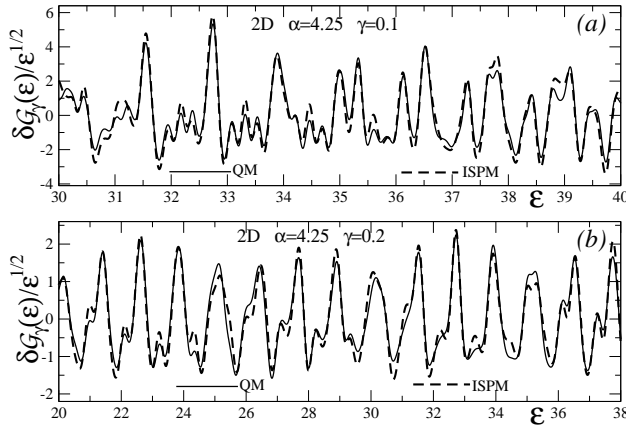


FIG. 6. The same as in Fig. 5 for $\alpha = 4.25$, but with the width parameters, $\gamma = 0.1$ (a), and $\gamma = 0.2$ (b).

POs (see Figs. 6 and 7 below for these bifurcations). For the gross shell structure ($\gamma \approx 0.2$ at $\alpha = 4.0$), only the shortest orbits (mainly a few shortest diameters) give the leading contributions. (This is in contrast to the 3D case where the circular orbits become also important; see Refs. [13, 18].) For instance, the gross shell structure in terms of the shortest POs for $\alpha = 6.0 - 7.0$ manifests at larger $\gamma \gtrsim 0.3$, unlike for the powers $\alpha = 4.0 - 4.25$. With decreasing the averaging parameter γ and increasing the power value α , the POs for larger scaled periods τ [or actions S , see Eq. (4.35)] become more significant [compare Figs. 4(b,d) and 7(b,d)]. In the case of the fine shell structure (e.g., $\gamma \approx 0.03$) the dominant contributions

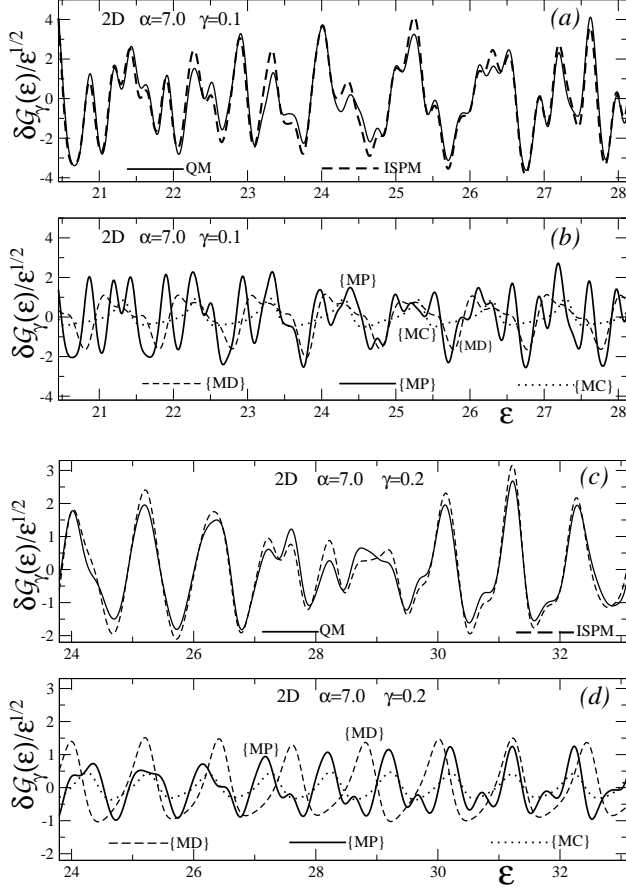


FIG. 7. The same as in Fig. 4, but for $\alpha = 7.0$ and other width parameters, $\gamma = 0.1$ (*a, b*) and $\gamma = 0.2$ (*c, d*).

are due to the bifurcating 2D $\mathcal{K} = 1$ POs [polygon-like POs denoted by $\{MP\}$; see Fig. 4(*b*)]. (This is similar to the situations in the elliptic [22] and spheroidal [11] cavities, and in the IHH potential [23].) However, the interference of much longer these one-parametric POs [such as $M(7, 3)$ for $\alpha = 4.0$ or $M(5, 2)$ for $\alpha = 6.0$] with a lot of the $M(2, 1)$ diameters explain some peaks, too. For smaller $\alpha = 4.0$ and 4.25 , the circle orbit contributions are not shown because they are not significant at these diffuseness parameters in the 2D case. (This is different from the 3D case; see Ref. [18] for the trace formulas based on the uniform approximation using the classical perturbation approach.) These contributions into the trace formula (4.46) are increasing functions of α , and they become significant at $\alpha \gtrsim 7$ even for the 2D case [see Fig. 7(*b, d*)]. An intermediate situation between the gross- and fine- shell structures, where all of POs become significant, are shown too at $\gamma = 0.1$ in Figs. 3 and 6,

and in Fig. 7(b,d) at $\gamma = 0.1$ and 0.2 . Note that our full analytical expressions (accessible for any long periodic orbits) for the classical PO characteristics at $\alpha = 4$ and 6 are quite useful in the simple ISPM calculations of the oscillating level density with a good accuracy up to the fine spectrum-structure resolutions by using, for instance, $\gamma \approx 0.03$ and 0.1 .

Figures 6 and 7 show a nice agreement of the fine-resolved semiclassical and quantum level densities $\delta\mathcal{G}_\gamma(\varepsilon)$ as functions of the scaled energy ε at the critical bifurcation points $\alpha = 4.25$ and 7.0 for the births of the star-like $(5, 2)$ and triangle-like $(3, 1)$ orbits, respectively.

Figures 8 and 9 show the scaled shell correction energies $\delta\mathcal{U}$ [Eqs. (4.52) for the semiclassical and (4.57) for the quantum results], normalized by the factor $\varepsilon_F^{-1/2}$, as functions of the particle number variable $N^{1/2}$. A good plateau is realized for the QM calculations of the scaled shell-correction energies [see the first equation in Eq. (4.57)] near the same averaging parameters $\tilde{\gamma}$ and curvature corrections as mentioned above. In the semiclassical calculations, the Fermi level ε_F is determined by the particle number conservation (4.53) with using the coarse-grained scaled-energy POT level density,

$$\mathcal{G}_{\gamma, \text{scl}}(\varepsilon) = \mathcal{G}_{\text{TF}}(\varepsilon) + \sum_{\mathcal{K}=0}^1 \delta\mathcal{G}_{\gamma, \text{scl}}^{(\mathcal{K})}(\varepsilon). \quad (5.1)$$

The oscillating ISPM components $\delta\mathcal{G}_{\gamma, \text{scl}}^{(\mathcal{K})}(\varepsilon)$ were determined above by Eqs. (4.46) and (4.38). We evaluated the Fermi level $\varepsilon_F(N)$ by varying the averaging width γ and found that there is no essential sensitivity within the interval of smaller γ ($\gamma \approx 0.1 - 0.2$). Moreover, even the TF density $\mathcal{G}_{\text{TF}}(\varepsilon)$ [Eq. (4.58) in Eq. (4.53) with $\mathcal{G}(\varepsilon) \approx \mathcal{G}_{\text{TF}}(\varepsilon)$] provides us a good value of ε_F in the POT calculations of the shell correction energies (4.52). The PO sums at $\alpha = 7.0$ converge for the shell correction density (4.46) by using the averaging width $\gamma = 0.2$ of a fine shell-structure resolution, and for the shell correction energies (4.52) with taking into account the same major simplest POs [about 4 repetition numbers ($M = 4$) for the circle and diameter orbits, and a few first simplest other $\mathcal{K} = 1$ (P) POs, such as $(3, 1)$, $(5, 2)$, $(7, 3)$ and $(8, 3)$; compare Figs. 9(c,d) with 7(c,d)]. For smaller diffuseness, $4 \lesssim \alpha \lesssim 6$, one has a similar PO convergence relation with the same $\gamma \approx 0.2$, but with much smaller contributions of the circular orbits. However, the dominating ($\mathcal{K} = 1$) PO families (P) are the $(5, 2)$, $(7, 3)$ and $(7, 3)$ POs at $\alpha = 4.25 - 6.0$ and 4.0 , respectively [see, for example, Figs. 8(a,b) and 4(c,d)]. As seen from Figs. 6, 7 and 9, we obtain a nice agreement between the semiclassical (ISPM, dashed) and quantum (QM, solid curve) results exactly at the bifurcations $\alpha = 4.25$ and 7.0 . Notice that the dominating contributions in these

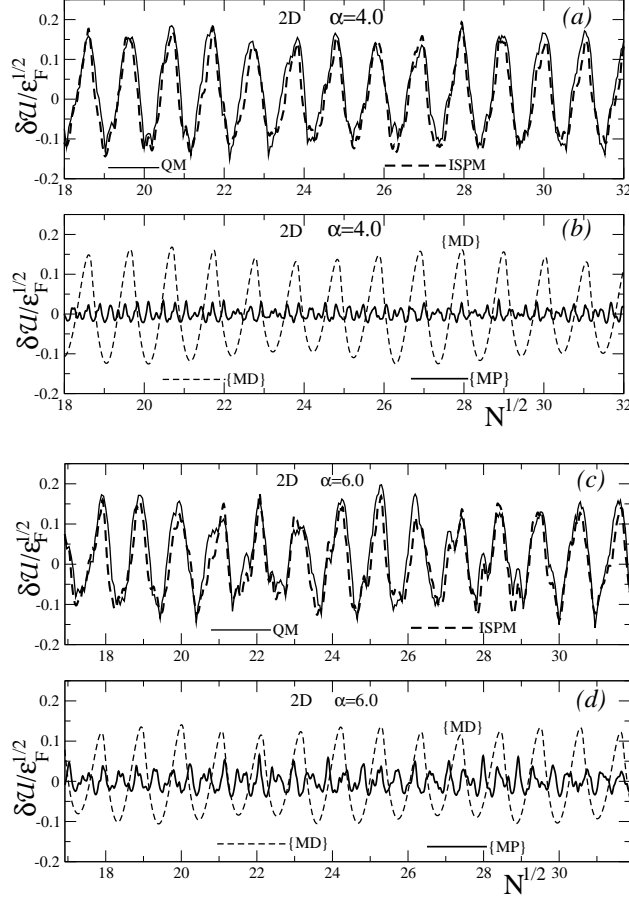


FIG. 8. Scaled shell correction energies $\delta\mathcal{U}$, normalized by the factor $\epsilon_F^{-1/2}$, as functions of square root of the particle number $N^{1/2}$ at the values of α , where the full analytical formulas are obtained for $\alpha = 4.0$ (a,b) and $\alpha = 6.0$ (c,d). *Panels (a,c)*: QM (solid curve) represents the quantum-mechanical results using the Strutinsky SCM, and ISPM (dashed curve) shows the semiclassical result using the TF approximation in the calculation of $N(\epsilon_F)$ by Eqs. (4.53) and (5.1). *Panels (b,d)*: the contributions of several POs into the shell correction energy $\delta\mathcal{U}$ are shown. Other notations are the same as in Figs. 4 and 7.

semiclassical results at the bifurcation point $\alpha = 7.0$ are coming from the interference of the bifurcating circle C and newborn $(3,1)$ orbits with the simplest diameters. As shown in Figs. 7(d) and 9(d), one can see more pronouncedly that the circle C and triangle-like $(3,1)$ orbits are mainly in phase, but the diameter $(2,1)$ is sometimes in phase to them and sometimes out of phase. Thus, the occurrence of a characteristic beating pattern in the level density amplitude at $\alpha = 7.0$ is due to the interference of the bifurcating orbits

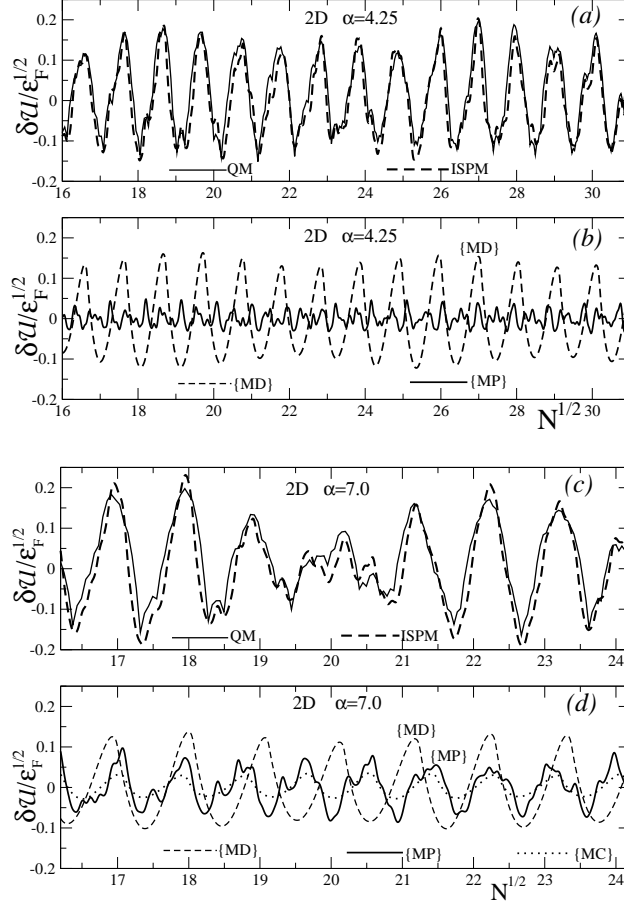


FIG. 9. The same as in Fig. 8; $\alpha = 4.25$ for the (5,2) bifurcation, and $\alpha = 7.0$ for the (3,1) bifurcation are shown in panels (a,b) and (c,d), respectively.

C and (3,1) with the shortest diameter (2,1) having all the amplitude of the same order in magnitude but different phases. The bifurcating circle $2C$ and star-like (5,2) orbits [as expected from the enhancement of the amplitudes of the circular C and triangular-like (3,1) POs in Fig. 2] are more important for $\alpha = 4.25$, though the primitive diameters become significant much compared to the bifurcation case $\alpha = 7.0$. The POs (3,1) and (5,2) yield more contributions near their bifurcation values of α , and even more on the right-hand side ($\alpha \gtrsim \alpha_{\text{bif}}$) in a wide region of α as mentioned above. The partner bifurcation orbits $\{C, (3,1)\}$ and $\{2C, (5,2)\}$, taken together with the simple diameter (2,1), give essential ISPM contributions of about the same order of magnitude and phase in Figs. 6, 7 and 9; as seen for example in Figs. 7(b,d) and 9(d) for the same $\alpha = 7.0$. The diameter ISPM contributions are close to the SSPM asymptotic ones near the bifurcation points $\alpha = 7.0$ and 4.25 (as for

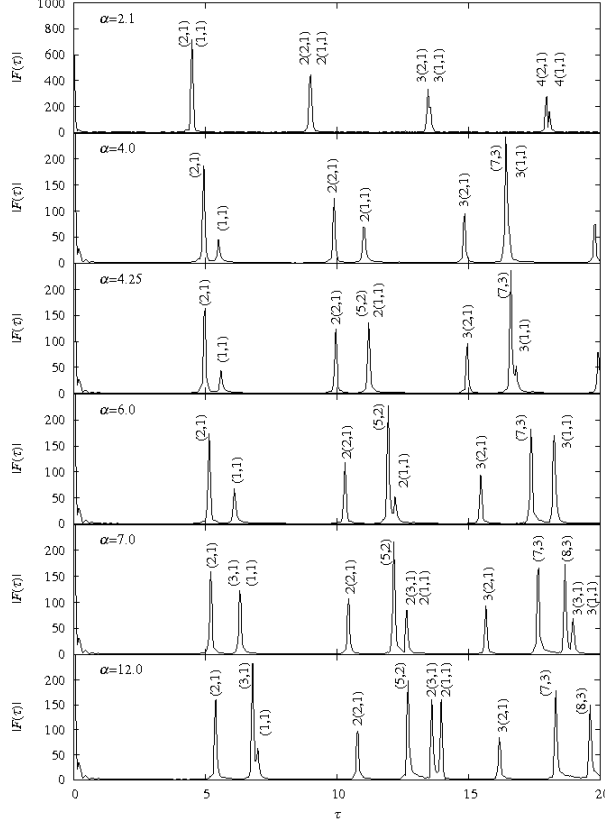


FIG. 10. Moduli of the Fourier transform $|F(\tau)|$ of the quantum scaled-energy level density (4.40) as functions of the dimensionless variable τ are plotted for several values of α ; MC and $M(n_r, n_\varphi)$ indicate the classical POs corresponding to each peak (see Fig. 1).

$\alpha = 4.0$ and 6.0) because they are sufficiently far from their single symmetry-breaking point at the harmonic oscillator value $\alpha = 2$.

Figure 10 shows the Fourier transform of the quantum-mechanical scaled-energy level density, see Eq. (4.40). For a smaller $\alpha = 2.1$, the diameter $(2, 1)$ orbit gives the dominant contribution to the gross-shell structure as the shortest POs; see the peak at $\tau \sim 5.0$. With increasing α , the amplitude of the circle orbit becomes again larger due to a prominent enhancement around the bifurcation point ($\tau \sim 6.2$ at $\alpha_{\text{bif}} = 7.0$). Notice that the newborn POs $(3, 1)$, $(5, 2)$, $(7, 3)$ and $(8, 3)$ give comparable contributions at $\alpha = 7.0$ [similarly, $(5, 2)$ and $(7, 3)$ for the bifurcation $\alpha = 4.25$] in nice agreement with the quantum Fourier spectra in Fig. 10. The contributions of the newborn triangle-like orbit family $(3, 1)$ having relatively a smaller scaled period $\tau_{(3,1)}$ and higher degeneracy $\mathcal{K} = 1$ become important and dominating for larger $\alpha \gtrsim \alpha_{\text{bif}} = 7$. The newborn $(3, 1)$ peak cannot be distinguished from the parent

circle C orbit near the bifurcation point α_{bif} as well as the diameter and circle orbits at α close to the HO limit, see Ref. [12]. We emphasize that the shell correction energies $\delta\mathcal{U}$ are similar to the oscillating parts of the level density averaged over the spectrum by using the Gaussian width $\gamma = 0.2$ at $\alpha = 4.0$ and 4.25 , which have mainly the gross-shell structure due to the shortest diameters. However, for this γ the fine-resolved shell structures (due to their interference with the other polygon-like and circular POs) are pronounced at larger powers near $\alpha = 6.0$, and especially, 7.0 .

Notice also that we do not show the numerical comparison of the ISPM trace formulas (4.46) and (4.52) vs quantum results for the HO limit $\alpha \rightarrow 2$ because, as shown analytically around Eq. (4.42), these formulas exactly analytically approach the corresponding HO trace formulas [13].

VI. CONCLUSIONS

We presented a semiclassical theory of quantum oscillations of the level density and energy shell corrections for a class of radial power-law potentials which turn out as good approximations to the realistic Woods-Saxon potential in the spatial region where the particles are bound. The advantage of the RPL potentials is that, in spite of its diffuse surface, the classical dynamics scale with simple powers of the energy, which makes it particularly easy for POT calculations, sometimes even completely analytically for $\alpha = 4$ and 6 , beyond the well known HO case $\alpha = 2$. The quantum Fourier spectra yield directly the contributions of the leading classical POs with the specific periods and actions into the trace formulas.

We described the main PO properties of the classical dynamics in the RPL potentials as the key quantities of the POT. Taking the simplest two-dimensional RPL Hamiltonian we developed the semiclassical trace formulae for any its power α , and studied various limits of α (the harmonic oscillator potential for $\alpha = 2$ and the cavity potential for $\alpha \rightarrow \infty$). The completely analytical results were obtained for the RPL powers $\alpha = 4$ and 6 . This can be applied for both 2D and 3D cases and allow us to far-going fine-resolved shell structures at $\gamma = 0.03 - 0.1$. This POT is based upon extended Gutzwiller's trace formula, that connects the level density of a quantum system to a sum over POs of the corresponding classical system. It was applied to express the shell correction energy δU of a finite fermion system in terms of POs. We obtained good agreement between the ISPM semiclassical and quantum-

mechanical results for the level densities and energy shell corrections at several critical powers of the RPL potentials. For the powers $\alpha = 4$ and 6 , we found also good agreement of the ISPM trace formulas with the SSPM ones. The strong amplitude-enhancement phenomena at the bifurcation points $\alpha = 7$ and 4.25 in the oscillating (shell) components of the level density and energy were observed in the remarkable agreement with the peaks of the Fourier spectra. We found a significant influence of the PO bifurcations on the main characteristics (oscillating components of the level densities and energy-shell corrections) of a fermionic quantum system. They leave signatures in its energy spectrum (visualized, e.g., by its Fourier transform), and hence, its shell structure. We have presented a general method to incorporate bifurcations in the POT, employing the ISPM based on the catastrophe theory of Fedoryuk and Maslov, and hereby, overcoming the divergence of the semiclassical amplitudes of the Gutzwiller theory and their discontinuity in the Berry&Tabor approach at bifurcations. The improved semiclassical amplitudes typically exhibit a clear enhancement near a bifurcation and on right side of it, where new orbits emerge, which is of the order $\hbar^{-1/2}$ in the semiclassical parameter \hbar . This, in turn, leads to the enhanced shell structure effects. Bifurcations are treated, again, in the ISPM leading to the semiclassical enhancement of the orbit amplitudes. The trace formulae are shown numerically to give good agreement with the quantum-mechanical level density oscillations for the gross- (larger coarse-graining averaging parameter γ and a few shortest POs), and the fine-resolved (smaller γ and longer bifurcating POs) shell structures, also for the shell-correction energies independent of γ . We found similar gross shell structures in the shell-correction energies with the corresponding averaged densities at $\gamma \approx 0.2$ for smaller powers $\alpha = 4 - 4.25$ of the gross shell structure, and for larger $\alpha \gtrsim 6$ of the fine shell structure. The fine-resolved and gross shell structures were found in the oscillating averaged densities at smaller averaging parameters $\gamma = 0.03 - 0.1$ and at larger $\gamma \gtrsim 0.2 - 0.3$, respectively. The fine-resolved shell structure for larger powers, $\alpha \geq 6$, occurs in a larger interval, $\gamma = 0.003 - 0.2$, including the essential contributions of the circle orbits along with the polygon-like and diameter orbits. Full explicit analytical expressions for the diameters and circle orbit contributions into the trace formula as functions of the diffuseness potential parameter α are specified too.

We may state that the semiclassical POT is well capable of explaining the main features of quantum shell structure in terms of a few classical periodic orbits for the RPL potentials. Bifurcations of POs are not simply an obstacle of the semiclassical theory, but they leave

clear signatures both in the quantum Fourier spectra and the locations of minima of the shell-correction energy δU which are plotted versus particle number and powers.

For prospectives, we intend a further study of shell structures in the 3D RPL potentials, applying both the ISPM and uniform approximations to treat the bifurcations, by varying continuously the power parameter α from 2 (harmonic oscillator) to ∞ (3D spherical billiards).

ACKNOWLEDGMENTS

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Appendix A: The stability factor, bifurcation powers and frequencies

Let us consider in more details the non-linear classical dynamics in the RPL Hamiltonian (2.1) for any real $\alpha \geq 2$. The critical values of the radial coordinate $r = r_C$ and angular momentum $L = L_C$ for the circle orbit (C) are determined by the solutions of the system of the two equations with respect to r and L :

$$\mathcal{F}(r, L) = 0, \quad \frac{\partial \mathcal{F}}{\partial r} = 0, \quad \text{where} \quad \mathcal{F}(r, L) \equiv p_r^2(r, L), \quad (\text{A1})$$

see Eq. (2.4). In the internal region where the stable orbits in the radial direction exist, one has a nonzero $\mathcal{F}_C'' = \partial^2 \mathcal{F}(r_C, L_C) / \partial r^2 = -4m E \alpha / r_C^2 < 0$ ($V_C'' > 0$). First equation in Eq. (A1) means that there is no radial velocity, $\dot{r} = 0$, and the next equation is that the radial force is equilibrating by the centrifugal force. For instance, for the potential (1.1), the solutions of the two these equations are the radius r_C and angular momentum L_C [15],

$$r_C = R_0 \left(\frac{2E}{(2+\alpha)E_0} \right)^{1/\alpha}, \quad L_C = p(r_C)r_C. \quad (\text{A2})$$

Using Eqs. (2.7) at $L = L_C$ and (A2) for r_C and L_C , one finds [15]

$$\omega_C = \omega_\varphi(L = L_C) = \sqrt{\frac{\alpha E_0}{m R_0^2}} \left(\frac{2E}{(2+\alpha)E_0} \right)^{1/2-1/\alpha}. \quad (\text{A3})$$

Applying now the second order expansion in $r - r_C$ to Eq. (2.4), one gets the first-order ordinary differential equation, which determines the radial CT $r(t)$ locally near the circle

PO $r = r_C$:

$$\begin{aligned} \dot{r} &= \pm \sqrt{\frac{\mathcal{F}_C''}{2m^2}} (r - r_C), \\ \mathcal{F}_C'' &= -2m \left[\frac{d^2}{dr^2} \left(V(r) + \frac{L^2}{2m r^2} \right) \right]_{r=r_C, L=L_C}. \end{aligned} \quad (\text{A4})$$

Integrating the dynamical equation in Eq. (A4), one obtains

$$\begin{aligned} r(t) &= r_C + (r' - r_C) \exp \left(\pm \sqrt{\frac{\mathcal{F}_C''}{2m^2}} t \right), \\ [r' &= r(t = t' = 0)] \quad . \end{aligned} \quad (\text{A5})$$

In the stable case, $\mathcal{F}_C'' < 0$ in Eq. (A5) for the CT $r(t)$ locally near the circle orbit $r = r_C$, one writes

$$r(t) = r_C + (r' - r_C) \exp(\pm i \Omega_C t), \quad (\text{A6})$$

where Ω_C is a positive radial frequency ω_r at $L = L_C$ [see Eq. (2.7)],

$$\Omega_C = \sqrt{|\mathcal{F}_C''/(2m^2)|} = \omega_r(L = L_C). \quad (\text{A7})$$

For the potential (1.1), this quantity is given by [15]

$$\Omega_C = \sqrt{\frac{\alpha(2+\alpha)E_0}{mR_0^2} \left[\frac{2E}{(2+\alpha)E_0} \right]} > 0. \quad (\text{A8})$$

From Eq. (A6) after the period T_C along the primitive circle orbit,

$$T_C = t'' - t' = t'' = \frac{2\pi n_\varphi}{\omega_C}, \quad \omega_C = \omega_\varphi(L = L_C), \quad (\text{A9})$$

one finds

$$\delta r'' \equiv r'' - r_C = \delta r' \exp(\pm i \Omega_C T_C), \quad \delta r' = r' - r_C. \quad (\text{A10})$$

The eigenvalues of the stability matrix \mathcal{M}_C for $M = 1$ in Eq. (2.9) are given by

$$\left(\frac{\partial r''}{\partial r'} \right)_{p_r'} = \exp(i \Omega_C T_C), \quad \left(\frac{\partial p_r''}{\partial p_r'} \right)_{r'} = \exp(-i \Omega_C T_C). \quad (\text{A11})$$

These two eigenvalues of the stability matrix are complex conjugated, as it must be, according to its general properties. So, as Ω_C is real [$\Omega_C > 0$, according to Eqs. (A7) and (A8)] the circle orbit is isolated stable PO. Substituting the expressions (A11) into the first equation in Eq. (2.9) and using Eqs. (A9) for the period T_C , (A3) and (A8) for the C-orbit frequencies ω_C and Ω_C , relatively, one obtains the last equation in Eq. (2.9) for the stability factor F_{MC} .

Appendix B: Scaling properties

For convenience, let us consider the classical dynamics in terms of the variables in dimensionless units $m = R_0 = E_0 = 1$. Due to the scaling property (2.2) for the classical dynamics in the Hamiltonian (2.1), the energy dependence of the action I_r [Eq. (2.5)], the angular momentum L , the frequency ω_r [Eq. (2.7)] and the curvature K [Eq. (2.12)] can be expressed in terms of the simple powers of the scaled energy ε ,

$$\varepsilon = E^{1/\alpha+1/2}. \quad (\text{B1})$$

In particular, one can express these classical quantities through their values at $\varepsilon = 1$ ($E = 1$),

$$\begin{aligned} I_i &= I_i(1)\varepsilon, \quad L = L(1)\varepsilon, \quad \omega_r^{-1} = \omega_r^{-1}(1)\varepsilon^{(2-\alpha)/(2+\alpha)}, \\ K &= K(1)/\varepsilon, \end{aligned} \quad (\text{B2})$$

where $I_i(1)$, $L(1)$, $\omega_r^{-1}(1)$, and $K(1)$ are the corresponding quantities of Eq. (B2), which are taken at “the scaled energy” $\varepsilon = 1$. Therefore, due to the scaling properties (2.2) and (B2), we need to calculate these classical dynamical quantities only at one value of the energy $\varepsilon = 1$. For simplicity of notations, we shall omit the argument $\varepsilon = 1$ everywhere, if it is not lead to misunderstandings.

The energy surface $I_r(L, E)$ for the potential (1.1) can be expressed explicitly in terms of the frequencies ω_r and ω_φ [see left identity in Eq. (2.8)],

$$I_r = \frac{2\alpha}{\alpha+2} \omega_r^{-1} - L f(L). \quad (\text{B3})$$

To prove this identity, we express Eq. (2.7) for ω_r^{-1} in terms of the determinant,

$$\omega_r^{-1} = \left(\frac{\partial I_r}{\partial E} \right)_L = \frac{\partial(I_r, L)}{\partial(E, L)} = \frac{\partial I_r}{\partial E} - \frac{\partial I_r}{\partial L} \frac{\partial L}{\partial E}. \quad (\text{B4})$$

Calculating directly the derivatives in this equation by using Eq. (B2), one obtains the expression for $\omega_r^{-1}(1)$. Solving then this equation with respect to $I_r(1)$, one arrives at Eq. (B3). Differentiating the identity (B3) term by term over L and using the definition for the ratio of frequencies $f(L)$ in Eq. (2.8), for the curvature (2.12) one finally obtains

$$\begin{aligned} K &= -\frac{2\alpha}{(\alpha+2)L} \frac{\partial \omega_r^{-1}}{\partial L} \\ &= -\frac{\alpha}{\pi(\alpha+2)L} \frac{\partial T_r}{\partial L}, \quad T_r = \frac{2\pi}{\omega_r}, \end{aligned} \quad (\text{B5})$$

where ω_r^{-1} is obviously simpler quantity to differentiate over L than $f(L)$, according to Eqs. (2.7) and (2.8) with the help of the textbook Ref. [29],

$$\omega_r^{-1} = \frac{1}{2\pi\sqrt{2}} \int_{x_{\min}}^{x_{\max}} \frac{dx}{\sqrt{Q(x, L, \alpha)}}, \quad (\text{B6})$$

$$Q(x, L, \alpha) = (1 - x^{\alpha/2}) x - L^2/2, \quad (\text{B7})$$

and $x = r^2$. The turning points $x_{\min}(L, \alpha)$ and $x_{\max}(L, \alpha)$ are determined by the following algebraic equation:

$$Q(x, L, \alpha) = 0. \quad (\text{B8})$$

Thus, we may calculate ω_r^{-1} and $f(L)$, and then, use Eqs. (B3) and (B5) for the radial action I_r , and curvature K at the scaled energy $\varepsilon = 1$. Then, one obtains their energy dependence through the scaling equations (B1) and (B2), respectively.

Appendix C: Full analytical classical dynamics for powers 4 and 6

For the powers $\alpha = 4$ and 6, the roots of function (B7), in particular, the turning points x_{\min} and x_{\max} can be obtained explicitly analytically. Therefore, one can find the explicit analytical expressions for the key quantities of the classical dynamics for the POT, namely, the radial frequency ω_r [Eq. (2.7)] (or the radial period T_r), and the frequency ratio $f(L)$ [Eq. (2.8)] in terms of the elliptic integrals from Ref. [29] (all in dimensionless units). Then, we calculate analytically also the curvature K through Eq. (B5) and action by Eq. (B3). For $\alpha = 4$, one has the cubic polynomial equation $Q(x, L, \alpha) \equiv x - x^3 - L^2/2 = 0$ [Eqs. (B7) and (B8)] for the three roots x_{\min} , x_{\max} and x_1 which are given by the Cardano formulas explicitly as functions of L in the physical region $L \leq L_C$, $r_1 < 0 \leq r_{\min} \leq r_{\max}$; $x_q = r_q^2$. For the radial period T_r [Eqs. (B5) and (2.7)], one obtains the analytical expression through these roots in terms of the complete elliptic integral $F(\pi/2, \kappa)$ of the first kind (see Refs. [18, 29]),

$$T_r = \frac{2\pi}{\omega_r} = \frac{\sqrt{2}}{\sqrt{x_{\max} - x_1}} F\left(\frac{\pi}{2}, \kappa\right), \quad (\text{C1})$$

where

$$\kappa = \sqrt{(x_{\max} - x_{\min}) / (x_{\max} - x_1)}. \quad (\text{C2})$$

For the ratio frequencies $f(L)$ [Eq. (2.8)], one finds

$$f(L) = \frac{L}{\pi\sqrt{2} x_{\max}\sqrt{x_{\max} - x_1}} \Pi\left(\frac{r_{\max} - r_{\min}}{r_{\max}}, \kappa\right), \quad (\text{C3})$$

where $\Pi(n, \kappa)$ is the complete elliptic integral of the 3rd kind [29].

For $\alpha = 6$, one has the polynomial equation of the 4th power, $Q(x, L, 6) \equiv x - x^4 - L^2/2 = 0$, having the 4 roots [two complex conjugated $x_1 + ix_2$ and $x_1 - ix_2$, and again, two real positive roots, x_{\min} and x_{\max} ; see Eqs. (B7) and (B8)]. The radial period T_r is determined through these roots by the expression [similar to Eq. (C1], see Refs. [18, 29],

$$T_r = \frac{\sqrt{2}}{\sqrt{AB(x_{\max} - x_1)}} F\left(\frac{\pi}{2}, \kappa\right), \quad (C4)$$

where

$$\kappa = \sqrt{\frac{(x_{\max} - x_{\min})^2 - (A - B)^2}{4AB}}, \quad (C5)$$

$$A = \sqrt{(x_{\max} - x_1)^2 + x_2^2}, \quad B = \sqrt{(x_{\min} - x_1)^2 + x_2^2}. \quad (C6)$$

(We reduced the 4-power polynomial equation to a cubic one and obtained its 4 analytically given roots, mentioned above, in the explicit Cardano's form as functions of L). For $f(L)$ of Eq. (2.8) at $\alpha = 6$, one obtains [29]

$$f(L) = \frac{\sqrt{2} L (A + B)}{\pi \sqrt{AB} (Ax_{\min} - Bx_{\max})} \times \left[\beta F\left(\frac{\pi}{2}, \kappa\right) + \frac{\beta - \beta_1}{2(1 - \beta^2)} \Pi\left(\pi, \frac{\beta^2}{1 - \beta^2}, \kappa\right) \right], \quad (C7)$$

where $\Pi(\varphi, n, \kappa)$ is incomplete elliptic integral of the 3rd kind,

$$\beta = \frac{Ax_{\min} - Bx_{\max}}{Ax_{\min} + Bx_{\max}}, \quad \beta_1 = \frac{A - B}{A + B}. \quad (C8)$$

The curvatures K [Eq. (B5)] for $\alpha = 4$ and 6 are determined by taking analytically the derivative of the radial period T_r [Eqs. (C1) and (C4)] over L through the derivatives of the roots $x_{\min}(L)$, $x_{\max}(L)$, $x_1(L)$ and $x_2(L)$ by using Eqs. (C2), (C5), (C6) and (C8), and the explicit expression for the derivative of $F(\pi/2, \kappa)$ over κ (see Ref. [29]). The expressions for the curvatures K at the both powers $\alpha = 4$ and 6 can be found in the closed analytical form through a rather bulky formulas, which contain the complete elliptic integrals of the 1st and 2nd kind (for instance, by using Mathematica).

Appendix D: Classical dynamics and boundaries for the diameters

For the primitive diameter $D = (2, 1)$, the action S_D (all in this appendix in dimensionless units) is specified analytically through the scaled period τ_D and energy ε by

$$S_D = \tau_D \varepsilon, \quad \tau_D = \frac{4\sqrt{2\pi}}{\alpha + 2} \Gamma\left(\frac{1}{\alpha}\right) \Gamma\left(\frac{1}{2} + \frac{1}{\alpha}\right), \quad (\text{D1})$$

where $\Gamma(x)$ is the Γ -function of a real positive argument x . For the diameter PO boundaries, one can use the same $L_- = 0$, but $L_+ = b_D L_C$, where

$$b_D = 1 - \frac{1}{2} \exp\left[-\left(\frac{L_D^{HO} - L_D}{2\Delta_D}\right)^2\right],$$

$$\Delta_D = \frac{1}{\sqrt{\pi M n_r K_D}} \quad (\text{D2})$$

(see Ref. [23] and more details in relation to the HO limit in Sec. IV D), $L_D = 0$ is the stationary point, $L_D^{HO} = L_C/2 = \varepsilon/(2\sqrt{2})$ is the upper angular momentum L_+ for the D orbits in the limit $\alpha \rightarrow 2$, in which $b_D \rightarrow 1/2$. In the semiclassical limit $\varepsilon \gg 1$, one has $b_D \rightarrow 1$. Δ_D is the Gaussian width of the transition region between these two asymptotic limits. The diameter-orbit curvature K_D for $\alpha \geq 2$ at the stationary point $L = L_D$ is given by

$$K_D = \frac{\Gamma(1 - 1/\alpha)}{\varepsilon \sqrt{2\pi} \Gamma(1/2 - 1/\alpha)}. \quad (\text{D3})$$

This exact analytical expression for the curvature K_D at any α was derived by using a power expansion in equation (B8) for the turning points, and in the identity (B3) over the variable proportional to L^2 near $L = 0$ up to the terms linear in L^2 . The Maslov phase for the diameter orbit was determined by Eq. (4.13) at $n_r = 2$ and $n_\varphi = 1$. Note that for the limit $\alpha \rightarrow 4$, the general expressions for the period τ_D and action S_D [Eq. (D1)], the Maslov index σ_D [Eq. (4.13)] with the same asymptotic (SSPM) limit of the constant part of the phase $\phi_d^{(D)} = -\pi(4b_D - 3)/4 \rightarrow -\pi/4$, and the curvature K_D [Eq. (D3)] for the diameter $(2, 1)$ are identical to those obtained earlier in Ref. [18].

Appendix E: The boundaries and curvature for circle orbits

For the arguments $\mathcal{Z}_p^{(\pm)}$ and $\mathcal{Z}_r^{(\pm)}$ of the error functions in Eq. (4.27), one originally has

$$\begin{aligned}\mathcal{Z}_{p,MC}^{(\pm)} &= \sqrt{-\frac{i}{2\hbar} \mathcal{J}_{MC}^{(p)} (p_r^{(\pm)} - p_r^*)}, \\ \mathcal{Z}_{r,MC}^{(\pm)} &= \sqrt{-\frac{i}{2\hbar} \mathcal{J}_{MC}^{(r)} (r^{(\pm)} - r_C)},\end{aligned}\tag{E1}$$

where $p_r^* = 0$ is the stationary point, $p_r^{(\pm)}$ and $r^{(\pm)}$ are maximal and minimal classically accessible values of p_r and r as the finite integration limits for the corresponding variables. To express the integration boundaries (E1) in an invariant form through the curvature K_C (4.30), and stability factor F_{MC} (2.9), one may use now the simple standard Jacobian transformations, and the definition of the angle variable Θ'_r as canonically conjugated one with respect to the radial action variable I_r by means of the corresponding generating function. In these transformations, we apply simple linear relations:

$$\begin{aligned}p_r'' - p_r^* &= \left(\frac{\partial p_r''}{\partial L} \right)^* (L - L^*), \\ r' - r^* &= \left(\frac{\partial r'}{\partial \Theta'_r} \right)^* (\Theta'_r - \Theta_r^*),\end{aligned}\tag{E2}$$

where we immediately recognize the Jacobian coefficients. Note that there is no crossing terms due to the isolated stationary point $I_r^* = 0$, $\Theta_r^* = 0$ and to equations for the canonical transformations. At the stationary point for the isolated circle PO, one has $f(L_C) = -(\partial I_r / \partial L)_{L=L_C} = -1/\sqrt{\alpha+2}$; see Eqs. (2.8) and (2.10). For the transformation of the derivative $\partial r'' / \partial \Theta'_r$, one can apply the Liouville conservation of the phase space volume for the canonical variables to arrive at $\partial r'' / \partial \Theta'_r = (\partial I_r / \partial L) / (\partial p'_r / \partial L)$. We use also the following identities:

$$\begin{aligned}F_{MC} &= -\mathcal{J}_{MC}^{(p)} \mathcal{J}_{MC}^{(r)} / \mathcal{J}_{MC} (p'_r, p''_r), \\ \frac{\partial p''_r}{\partial p'_r} &= \frac{\partial p''_r / \partial L}{\partial p'_r / \partial L} = \frac{r' p'_r}{r'' p''_r} = 1,\end{aligned}\tag{E3}$$

i.e., $|\mathcal{J}_{CT}(p'_r, p''_r)| = 1$ at the PO conditions $r' \rightarrow r'' \rightarrow r_C$, $p'_r \rightarrow p''_r \rightarrow 0$. After such simple algebraic transformations, one obtains Eq. (4.29) for the arguments of the error functions in Eq. (4.27).

The expression (4.30) for the circle-orbit curvature K_C (in dimensionless units at $\varepsilon = 1$) was obtained from expansion of $f(L)$ [see Eq. (2.8)] as function of L in powers of $L_C - L = \epsilon^2$

up to the 2nd order terms in ϵ . For this purpose, we have to solve first Eq. (B8) for the turning points r_{\min} and r_{\max} , which are the integration limits in Eq. (2.8) by using standard perturbation theory looking for the solutions in the following general form (r is taken below in units of R_0),

$$\begin{aligned} r_{\max} &= r_C + c_1\epsilon + c_2\epsilon^2 + c_3\epsilon^3 + c_4\epsilon^4 + \dots, \\ r_{\min} &= r_C - c_1\epsilon + c_2\epsilon^2 - c_3\epsilon^3 + c_4\epsilon^4 + \dots. \end{aligned} \quad (\text{E4})$$

Existence of such form of the solutions follows from a symmetry of the equation (B8) with respect to the change of the sign of ϵ . Substituting these solutions into Eq. (B8) for arbitrarily small ϵ , one gets the system of the recurrent equations for the coefficients c_n . The solutions of this system up to the 4th order in a perturbation parameter ϵ is given by

$$\begin{aligned} c_1 &= \sqrt{\frac{L_C}{\alpha}}, \quad c_2 = -\frac{\alpha+1}{6}c_1^2, \quad c_3 = \frac{(\alpha-2)(2\alpha+5)}{72}c_1^3, \\ c_4 &= -\frac{(\alpha+1)(4\alpha^2+8\alpha+13)}{1080}c_1^4, \end{aligned} \quad (\text{E5})$$

and so on. We transform now the integration variable r in the integral of Eq. (2.8) for $f(L)$ to y , $r = r_C(1-y)$, so that

$$f(L) = -\frac{L-\epsilon^2}{\pi} \int_{y_{\min}}^{y_{\max}} \frac{dy}{(1-y)\sqrt{Q(y, L, \alpha)}}, \quad (\text{E6})$$

where $Q(y, L, \alpha)$ is given by Eq. (B7),

$$\begin{aligned} Q(y, L, \alpha) &= 2r_C^2 \left[1 - \frac{2}{\alpha+2}(1-y)^\alpha - L_C^2 + 2L_C\epsilon^2 - \epsilon^4 \right] \\ &\equiv (y_{\max} - y)(y - y_{\min})\mathcal{R}(y). \end{aligned} \quad (\text{E7})$$

Here,

$$\begin{aligned} y_{\max} &= \bar{c}_1\epsilon - \bar{c}_2\epsilon^2 + \bar{c}_3\epsilon^3 - \bar{c}_4\epsilon^4, \\ y_{\min} &= -\bar{c}_1\epsilon - \bar{c}_2\epsilon^2 - \bar{c}_3\epsilon^3 - \bar{c}_4\epsilon^4, \end{aligned} \quad (\text{E8})$$

where $\bar{c}_n = c_n/r_C$. We use the last representation in Eq. (E7), introducing a new function $\mathcal{R}(y)$ of the new variable y to separate the singularities of the integrand in Eq. (E6) due to

the turning points. This integrand has to be integrated exactly by using a smooth function $\mathcal{R}(y)$ of y , which can be expanded in y at $y = 0$ up to the second order,

$$\mathcal{R}(y) = \mathcal{R}(0) + \mathcal{R}'(0)y + \frac{1}{2}\mathcal{R}''(0)y^2 + \cdots. \quad (\text{E9})$$

In order to get analytically the final result, we note that y in this expansion is of the order of ϵ , according to Eq. (E8). Substituting then these expansions (E8) and (E9) into very right of Eq. (E7), we expand their middle in y at $y = 0$ up to the 4th order. After the cancellation of ϵ^2 from both sides, and simple algebraic transformations, one has

$$\begin{aligned} \mathcal{R}(0) &= \frac{2L_C}{\bar{c}_1^2} [1 + \epsilon^2 \bar{c}_2 (1 - k_2)] + \mathcal{O}(\epsilon^4), \quad k_2 = \frac{\bar{c}_2}{\bar{c}_1^2}, \\ \frac{\mathcal{R}'(0)}{\mathcal{R}(0)} &= 2k_2 + \mathcal{O}(\epsilon^2), \quad \frac{\mathcal{R}''(0)}{\mathcal{R}(0)} = 2k_2(3k_2 + 1) + \mathcal{O}(\epsilon^2). \end{aligned} \quad (\text{E10})$$

For the calculation of the circle orbit curvature K_C , we obviously need only quadratic terms in ϵ [linear in $(L_C - L)$]. Therefore, one may neglect the ϵ^2 corrections in the second and third lines of Eq. (E10) because they are multiplied by $y \sim \epsilon$ and $y^2 \sim \epsilon^2$ in the expansion (E9), respectively. Substituting now expansions (E8) and (E9) into the integral over y in Eq. (E6), and taking $\mathcal{R}(0)$ off the integral, one then expands to the second order all quantities of the integrand in $y \sim \epsilon$, except for $(y_{\max} - y)(y - y_{\min})$ under the square root (in the denominator) which can be integrated exactly. Taking remaining integrals as $\int dy y^n / \sqrt{(y_{\max} - y)(y - y_{\min})}$ from y_{\min} to y_{\max} [see Eq. (E8)], and then, expanding finally $f(L)$ [Eq. (E6)] in ϵ , we find that the linear terms exactly disappear. It must be the case because $f(L)$ is an even function of ϵ . Thus, the coefficient in front of ϵ^2 with the expressions for \bar{c}_n ($n = 1, 2, 3$) from Eq. (E5) is just Eq. (4.30) for the curvature K_C . We can also use this perturbation method for the calculation of the next order curvatures, for instance, the third derivative of I_r over L , which appears in expansion of the phase integral in the exponent up to the third order terms near the stationary points within a more precise (3rd-order) ISPM [11].

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